

# An Analytical Model of the Fisher Equation with Memory Functions

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## Abstract

Numerous empirical studies have documented that increases in inflation have negative, short-run but positive, long-run effects on stock returns. The former transitory inflation anomaly has been explained by a number of theories, while the latter permanent inflation response to an increase in inflation is consistent with the well-known Fisher effect. In this paper we derive an analytical model of the tax version of the Fisher equation that incorporates a memory function for stock prices and inflation rates based on fractional calculus. Importantly, this model demonstrates that *both* negative, short-run and positive, long-run effects are possible in response to an increase in inflation. We conclude that the Fisher equation is more general than previously believed upon introducing memory into its parameters. Also, our results imply that memory functions are likely useful mathematical operators in other areas of financial economics wherein variables have time dependent behavior.

## I. Introduction<sup>2</sup>

Now-famous work by Fisher (1930) showed that inflation has a positive effect on bond returns. To demonstrate this relationship Fisher invented distributed lag regression models, which revealed that increases in inflation at the turn of the 20<sup>th</sup> century could have gradually diminishing but long-term effects on bond returns up to 25 years later. Formally stated, the familiar Fisher equation is  $i_n = i_r + \mathbf{p}$ , where  $i_n$  = the nominal interest rate,  $i_r$  = the real interest rate, and  $\mathbf{p}$  = the expected inflation rate. Extending this so-called *Fisher effect* to stock returns, some empirical studies (e.g., see Jaffe and Mandelker (1976) and Boudoukh and Richardson (1993)) have found that stock returns are also positively related to inflation in the long run. By consensus, this body of empirical research suggests that capital markets have a long memory concerning inflation shocks and that current capital prices are a diminishing function of prior inflation forces.

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<sup>2</sup> Glossary of Parameters and other variables:

$\mathbf{p}$	rate of inflation
$i_n$	nominal interest rate
$i_r$	real interest rate
$T$	marginal tax rate on investment income
$k$	stock price
$s$	stock price in logarithmic form $s = \log k$
$r$	real stock return rate
$u$	LaPlace Transform (LT) variable
$w$	frequency
$t$	time

All the parameters are a function of time, with the exception of the constant term  $T$ .

However, considerable controversy has surrounded the fact that many researchers (e.g., see Lintner (1975), Bodie (1976), Fama and Schwert (1977), Nelson (1976), Schwert (1981), Geske and Roll (1983), and others) have found that inflation has a negative, short-run impact on stock returns. Various theories have been proposed to explain this anomalous stock return behavior, including a money demand channel by Nelson (1976), incorrect discounting of earnings by Modigliani and Cohn (1979), tax treatment of inventory and depreciation by Feldstein (1980) and Summers (1981), federal budget deficits by Geske and Roll (1983), monetary policy by Pearce and Roley (1985), and real wealth effects by Stulz (1986). An excellent overview of extant empirical and theoretical studies can be found in Ely and Robinson (1997). Also, a recent paper by Anari and Kolari (1999) applied vector autoregression analyses (VAR) to show that, while inflation shocks do have a permanent positive effect on stock prices in the long run, their initial, transitory response is negative. More specifically, the authors reported impulse response functions that demonstrate how unexpected increases in inflation have initial negative effects on stock price indexes (for about two years) in seven large industrial countries and thereafter a positive, permanent effect on stock price indexes gradually occurs (over the next 10 to 20 years).

Unlike previous theoretical work, the present paper re-specifies the Fisher relation as a dynamic process with time-dependent variables. Following long-run stock return studies cited above, we focus on Darby's (1975) tax version of the Fisher equation, or  $i_n = (i_r + p) / (1 - T)$ , where  $T$  is the tax rate. Our model transforms the relationship between nominal interest rates and expected inflation rates by specifying common stock prices, the rate of inflation, and the real rate of return as functions of time that reflect memory of their past levels. For example, if the inflation process is described by a linearly increasing function over time, a negative memory factor ( $\mathbf{b}$ ) would mean that previous knowledge of inflation cycles by market participants would result in lowered expectations of future inflation (i.e., expected inflation at time current  $t$  is below actual inflation at  $t$ ). Conversely, a positive memory factor ( $\mathbf{b}$ ) would add to expectations of future inflation above actual current levels. Thus, expected inflation is conditional on the current level of inflation in addition to market participants' memory of previous information about inflation. We assume that the tax rate remains constant. Each participant in the financial markets has the same memory of the past and the same understanding of how it affects future values of the variables. Each participant remembers or evaluates the most recent experience as being the most relevant, and so the memory functions place more weight on more recent events.

We begin by developing the model in the general case of a time-varying real interest rate, and then simplify by assuming that the real rate is constant as in Fisher. We investigate the behavior of the variables in the model by introducing an inflation shock that diminishes over time. We find that both short-run declines in stock prices as well as long-run increases in stock prices are produced in response to an unexpected increase in inflation using this dynamic approach to inflation in the Fisher relation.

Our analytical results are consistent with empirical evidence, which has documented that stock returns respond negatively to an increase in inflation in the short run but positively in the long run. Importantly, they imply that the Fisher equation is more general than previously believed upon introducing memory into its parameters. Both short-run and long-run responses to inflation shocks are possible when nominal interest rates and inflation rates are considered dynamic variables, in contrast to previous research that does not specify the variables in terms of memory functions. Previous research has supposed that the initial negative response of stock returns to inflation shocks is anomalous due to the fact that nominal returns and inflation rates should be positively related to one another under the Fisher effect. However, our results show that this negative response is part of a longer-run process that eventually becomes positive over time. A secondary but nonetheless important contribution of the present study is to extend mathematical memory functions from the physical sciences to the social science fields of finance and economics. These functions may well be applicable to a variety of

problems that would benefit from taking into account time dependent behavior among the variables under study. For example, if there is momentum in stock prices over long periods of time of three to five years, as proposed by behavioral finance theorists in recent years (e.g., see Chan, Jagadeesh, and Lakonishok (1996, 1998) and many others cited therein), memory functions could be useful in modelling stock price behavior. The next section specifies the Fisher relation, section 3 re-specifies this relation with time-dependent variables, section 4 demonstrates how a sudden increase in inflation that gradually diminishes over time affects stock prices, and section 4 gives the summary and conclusion.

## II. Background Discussion

Following Darby (1975), we can rewrite the Fisher equation for stocks as:

$$s(t + dt) - s(t) = (r + \mathbf{p}) / (1 - T), \quad (1)$$

where  $s$  = stock price in log form (or log  $k$ ),  $r$  = expected real stock return, and other terms are as before. This standard equation does not account for the initial, negative reactions of stock prices to inflation that the empirical evidence has revealed. In this respect we consider hidden memory in  $s(t)$ ,  $r(t)$ , and  $\mathbf{p}(t)$  not specified explicitly in equation (1). Also, the equation above typically assumes that the real stock return  $r$  is constant over time. Here we begin by allowing  $r$  to be time dependent and then later restrict  $r$  to be constant.

The basic notion of memory functions is widely recognized in the physical sciences. For example, the evolutionary theory of Darwin is based on the idea that living organisms have memory of past generations embedded in genetic material. Another example is that of an exploding population destroying its food supply faster than it reproduces itself (see Volterra (1930)). Eventually the supply of food is insufficient to meet demand and depopulation occurs due to starvation. This cycle reverses itself over time as food production increases at some point. Also, numerous applications of memory functions have been published in the field of physics (see Graffi (1936), Sposito (1980), Bagley and Torvik (1983, 1986), Le Mehaute and Crépy (1983), Jacquelin (1984, 1991), Körnig and Müller (1989), Hu and Cushman (1994), Caputo and Plastino (1998), and Caputo (2000) and others).

While memory functions are fairly common in the physical sciences, scant work is available on their application in the field of social sciences. In this regard, it seems reasonable to assume that individual agents act and react to situations based on personal or learned experience. Also, it is natural that individuals place more weight on recent events relative to more distant experiences in their behavior. That is, the informational content of variables employed in responding to events diminishes in value over time. In this regard, a seminal paper by Caputo (1998) incorporated memory functions in Keynesian equilibrium equations of the macroeconomy and demonstrated that steady state conditions can be achieved at a monotonically decreasing rate in the long run.

Here we seek to introduce into the Fisher equation a memory mechanism, which simulates instantaneous reactions of the system to sudden changes in inflation and subsequent delayed effects. To begin we write equation (1) in differential form. Assuming  $dt$  unitary, if the differential  $dk$  of stock price ( $k$ ) may be considered infinitesimal in the interval  $dt$ , which is generally realistic, we may rewrite equation (1) as:

$$ds(t) / dt = (r + \mathbf{p}) / (1 - T) \quad (2)$$

### III. A Modified Fisher Equation with Memory

Departing from previous Fisher equation analyses, we assume that the parameters appearing in equation (2) are mathematical operators defined as a function of time, with the exception of a constant tax rate. That is, we assume that market participants in the stock market behave as if they had memory of past movements in stock prices, real rates of return, and the inflation rate. Of course, this assumption is realistic to the extent that the appropriate memory weights are used for these variables.

The mathematical memory mechanism employed is a fractional derivative of fractional order  $n$  (see Caputo (1969) and Lucko and Gorenflo (1998)) defined as:

$$\mathcal{I}^n f(t) / \mathcal{I}t^n = (1 / (\Gamma(1-n))) \int_0^t (df(\tau) / d\tau) d\tau / (t-\tau)^n, \quad (3)$$

where  $n$  is subject to the limits  $0 < n < 1$ , and  $\Gamma$  is the gamma function. In practice the derivative of fractional order  $f(t)$  is constructed with a weighted mean of the first order derivative  $[df(\tau)/d\tau]_v$  in the time interval  $[0, t]$ , which is a sort of feedback system. That is, the values of  $[df(\tau)/d\tau]_v$  at time  $v$  far apart from  $t$  are given smaller weight than those at times  $v$  closer to  $t$ . Hence, the weights are increasingly smaller with increasing time separation from the time  $t$  to imply that the effect of the past is fading with increasing time. When  $n = 0$  and  $f(0) = 0$ , the fractional order derivative reduces to the functions themselves. Importantly, the weights multiplying the first order derivative of  $f(t)$  inside the integral appearing in equation (3) can be chosen in many ways. The definition adopted in equation (3) is appropriate because it is algebraically simple, allows easy solutions, and has commonly been applied in previously cited scientific studies. We begin with this specification of the Fisher equation for purposes of generality, but later introduce the restriction that  $r$  is constant, which is commonplace in financial economics literature.

Modifying equation (3) to introduce a memory mechanism which acts on  $r$ ,  $ds(t)/dt (= s)$ , and  $p$ , we obtain the following initial model:

$$I s + m \mathcal{I}^n s / \mathcal{I}t^n = C \left[ ar + b \mathcal{I}^n r / \mathcal{I}t^n + \alpha p + b \mathcal{I}^n p / \mathcal{I}t^n \right], \quad (4)$$

where the parameters  $C = 1 / (1 - T)$ ,  $I$ ,  $C$ ,  $a$ , and  $\alpha$  are dimensionless, and parameters  $b$ ,  $m$  and  $\beta$  have dimension time to the power  $n$ . In equation (4) the memory acts on a separate amount of the function – namely, the fraction represented by  $I/m$  for  $s$ ,  $b/a$  for  $r$  and  $\beta/a$  for  $p$ . The presence of all parameters  $a$ ,  $b$ ,  $\alpha$ ,  $\beta$ ,  $I$ , and  $m$  implies the possibility that only a proportional part of the variables  $s$ ,  $r$ , and  $p$  enters with memory in the modified Fisher equation. It is noteworthy to observe how the memory functions capture the past. What the fractional derivative memory functions are remembering is their past values as defined by equation (3), which implies that the function is constructed by adding to the initial value the successive weighted increments over time. The increments per unit time are represented by the first order derivative under the integral sign, and the weights are represented by the factor of the first order derivative in equation (3) which are decreasing with increasing time separation from the time  $t$ . Thus, a variable's value is a weighted mean of its past values. Referring to equation (4), if the inflation rate  $p(t)$  at the beginning of the year is zero, the inflation rate at the time  $t$  consists of a part proportional to  $\alpha$ , which is the actual rate at the time  $t$ , plus a part proportional to  $\beta$ , which enters the equation due to the memory effect. In general, it is assumed that  $\alpha = 1$  and  $|\beta| < 1$ . The same logic applies to the other variables appearing in equation (4).

With these definitions in hand, equation (4) is an integro-differential equation which we may solve using the LaPlace Transform (LT) (e.g., see Caputo (1984)) to get:

$$\begin{aligned} \mathbf{I} u S - \mathbf{I} s(0) + \mathbf{m} u^{1+n} S - \mathbf{m} s(0) u^n - \dot{\mathbf{m}} s(0) u^{n-1} = \\ C \left[ aR + \mathbf{a}P + b u^n R + \mathbf{b} u^n P - b u^{n-1} r(0) - \mathbf{b} u^{n-1} p(0) \right] \end{aligned} \quad (5)$$

where  $u$  is the LT variable and capital letters indicate the LT of the corresponding lower case letters. To solve equation (5) we isolate the LaPlace Transform of stock price ( $s$ ) or  $S$ :

$$\begin{aligned} S = s(0) / u + \dot{\mathbf{m}} s(0) u^n / u^2 (\mathbf{I} + \mathbf{m} u^n) + \\ + C \left[ (aR + \mathbf{a}P) / u (\mathbf{I} + \mathbf{m} u^n) \right] + C \left[ (bR + \mathbf{b}P) u^n / u (\mathbf{I} + \mathbf{m} u^n) \right] + \\ - C \left[ (br(0) + \mathbf{b}p(0)) u^n / u^2 (\mathbf{I} + \mathbf{m} u^n) \right] \end{aligned} \quad (5')$$

In order to find the solution  $s$ , we must find the inverse LaPlace Transform ( $LT^{-1}$ ) of equation (5'), which is derived in the Appendix A for each of the terms shown in brackets. Using also equation (A3) in Appendix A, we get:

$$\begin{aligned} s = s(0) + \left\{ \mathbf{m} \left[ \dot{s}(0) + \right. \right. \\ \left. \left. - (C / \mathbf{I})(ar + \mathbf{a}p) \right] + C(br + \mathbf{b}p) + \right. \\ \left. - C(br(0) + \mathbf{b}p(0)) \right\} * \mathbf{j}_2(\mathbf{I}, \mathbf{m}, t) + (C / \mathbf{I}) \int_0^t (ar + \mathbf{a}p) dt \end{aligned} \quad (5'')$$

where  $\mathbf{j}_2(\mathbf{I}, \mathbf{m}, t)$  (with the dimension of time to the power  $-n$ ) is the monotonic decreasing function given in equation (A1) of the appendix.

The study of the effect of memory in the frequency domain can be derived by substituting  $u$  with  $i\omega$  in the previous formulae. It can be easily seen from equation (A6) in the appendix that the effect is a low pass filtering which implies a relative transfer of energy from high to low frequencies, which tends to stabilise the system (in fractal language).

#### IV. Effects of an Impulse Memory Functions

We next obtain the memory parameters appearing in the above formulae which one may use to estimate the future effects of inflation on stock prices using empirical data. Since the economy can be characterised as a dynamic system, the appropriate analytical framework is comparable to that utilised in studying inelastic materials (see Bagley and Torvik (1983, 1985)) and porous media (see Caputo (1999)). In this instance we excite the system with an impulse signal of inflation and then recover the memory parameters from the recorded reaction of the system fit to the analytical model. As such, we are interested in the responses of the parameters of the Fisher equation to unexpected movements or shocks in inflation. To model inflation shocks we will consider the case of an impulse pattern of inflation over time. Here we assume that a sudden increase in inflation gradually vanishes over time. While we derive results for an impulse memory function, the equations can be easily adapted for step and box functions to simulate alternative time paths of inflation shocks.

To simplify matters, as in the original Fisher equation wherein  $a = \lambda = \alpha = 1$ , we impose the restriction that  $r$  is constant and that the impulse of inflation occurs at time  $t_0$  according to the following expressions:

$$\mathbf{p} = c \quad \text{for } t < t_0 \quad (6)$$

$$\mathbf{p} = c + \mathbf{g}d(t - t_0) \quad \text{for } t \geq t_0 \quad (6')$$

which are substituted into equation (5''). Using the formulae in Appendix A, we get:

$$s = s(0) + \dot{\mathbf{m}}s(0) * \mathbf{j}_2(1, \mathbf{m}, t) + C \left\{ (r + c) * \mathbf{j}_3(1, \mathbf{m}, t) + \right. \\ \left. + \mathbf{g}\mathbf{j}_3(1, \mathbf{m}, t - t_0) + \mathbf{b}\mathbf{g}\mathbf{j}_2(1, \mathbf{m}, t - t_0) \right\} \quad \text{for } t_0 < t \quad (7)$$

$$s = s(0) + \mathbf{m} \left[ \dot{s}(0) - (r + c) \right] \mathbf{j}_4(1, \mathbf{m}, t) + C(ar + \mathbf{a}c)t \quad \text{for } t < t_0. \quad (7')$$

Removing the linear effect of the constant real rate of capitalization and inflation rate,  $C(r + c)t$ , and again using the formulae in Appendix A, we obtain finally

$$s = s(0) + \mathbf{m} \left[ \dot{s}(0) - C(r + c) \right] \mathbf{j}_4(1, \mathbf{m}, t) + \mathbf{g}C + C\mathbf{g} \left\{ \mathbf{b} - \mathbf{m} \right\} \mathbf{j}_2(1, \mathbf{m}, t - t_0) \quad \text{for } t_0 < t \quad (8)$$

$$s = s(0) + \mathbf{m} \left[ \dot{s}(0) - C(r + c) \right] \mathbf{j}_4(1, \mathbf{m}, t) \quad \text{for } t < t_0.$$

(8')

When  $\mathbf{g} = 0$  equation (8) coincides with (5'') of section III. In this case the effect on stock prices of the inflation impulse is

$$C\mathbf{g} \left[ 1 + \left\{ (\mathbf{b} - \mathbf{m}) / \mathbf{m} \right\} \mathbf{j} \left( (t - t_0) / \mathbf{m}^{1/n} \right) \right]$$

(9)

where  $\mathbf{j} \left( (t - t_0) / \mathbf{m}^{1/n} \right)$  is a function of the time measured in units of relaxation time  $\mathbf{m}^{1/n}$  (see equation (A1) in Appendix A). Equation (9) converges monotonically to  $C\mathbf{g}$  at  $t = \infty$ .

Regarding the parameters in equation (9), the relaxation time and the values of  $\mathbf{m}$  and  $\mathbf{b} - \mathbf{m}$  of the system may be retrieved from observations of stock prices fit to equations (8) and (8').

Since  $\mathbf{j} \left( (t - t_0) / \mathbf{m}^{1/n} \right)$  is a function decreasing from 1 (at  $t = 0$ ) to 0 (at  $t = \infty$ ), as shown in equation (9), the effect of an inflation impulse on stock prices is a decreasing or increasing function depending on the sign of  $\mathbf{b} - \mathbf{m}$ .

At  $t = t_0$  the value of equation (9) is  $C\mathbf{g}\mathbf{b}/\mathbf{m}$  and, if  $\mathbf{b} - \mathbf{m} < 0$ , the values of equation (9) increase asymptotically to  $C\mathbf{g}$ . Given that  $\mathbf{j}_4(1, \mathbf{m}, t)$  in equation (8) is almost a linear function of  $t$ , it is clear from equation (9) that, when  $\mathbf{b} - \mathbf{m} < 0$ , after an initial change in stock prices at time  $t = t_0$ , subsequent changes in stock prices will asymptotically reach the value  $C\mathbf{g}$  with increasing time as  $\mathbf{j}_2(1, \mathbf{m}, t)$  is vanishing. Importantly, the values of the parameters appearing in equations (8) and (8') can be adjusted to fit the anomalous negative, short-run but positive, long-run reactions of stock price to an inflation shock.

So far we have demonstrated that the initial response of stock prices to a positive inflation impulse as defined in equation (5'') is governed by the factor  $\mathbf{b} - \mathbf{m}$  of  $\mathbf{j}_2(1, \mathbf{m}, t)$ . The maximum value of the amplitude of the stock price response is  $C\mathbf{g}\mathbf{b}/\mathbf{m}$  at  $t = t_0$ , which is negative when  $\mathbf{b}/\mathbf{m} < 0$ , in which case the function gradually increases and eventually reaches the positive value  $C\mathbf{g}$ . That is, in order to cause an initial negative reaction to the positive impulse of inflation, either  $\mathbf{b}$  or  $\mathbf{m}$  must be negative. The duration of the negative response of the prices of stocks to the positive impulse of inflation may be estimated from equation (9). The relation between this duration and the ratio  $\mathbf{b}/\mathbf{m}$  is shown in Figure 1, where the time is measured in units of the relaxation time  $\mathbf{m}^{1/n}$  and the ratio  $\mathbf{b}/\mathbf{m}$  can be interpreted as the ratio of the amplitudes of the memory mechanisms operating on  $s$  and  $\pi$ , respectively. Here we see that the duration of initial negative response of stock prices is a decreasing function of  $\mathbf{b}/\mathbf{m}$  (e.g., when  $\mathbf{b}/\mathbf{m} = -1.1$ , the duration measured in units of relaxation time is about 0.7 for

all values of  $n$ ). Since  $m^{1/n}$  is the relaxation time and  $n$  is not integer, then  $m > 0$  and  $b < 0$ . On the other hand, when  $b/m > 0$ , the stock price response to a positive impulse of inflation is positive and decreases asymptotically to  $Cg$

We also note that, when the impulse of inflation is negative (deflation), then  $g < 0$  and all the effects change sign. That is, a negative shock in inflation will produce a positive, short run but negative, long-run effect on stock prices (and returns).

Finally, we note that if  $r$  varies in time, its effect appears to be smoothed by the memory in  $s$  and is not affected by the memory in  $\pi$ .

## V. Summary and Conclusions

Numerous empirical studies have documented that increases in inflation have negative, short-run but positive, long-run effects on stock returns. The former transitory inflation anomaly has been explained by a number of theories, while the latter permanent inflation response to an increase in inflation is consistent with the well-known Fisher effect. In this paper we used fractional calculus to derive an analytical model of the Fisher equation for stocks in which memory functions for stock prices, real rates of interest, and inflation are introduced. The memory functions reflect the notion that market participants in the stock market employ information about past movements of these variables in determining current stock prices. In this respect past events are assigned progressively lower weights to model the past as gradually fading in market participants' memory. Current expected inflation is conditional on actual inflation in addition to market participants' memory of previous information about inflation. For example, if the inflation process is described by a linearly increasing function over time, a negative memory factor ( $\mathbf{b}$ ) would mean that previous knowledge of inflation cycles by market participants would result in lowered expectations of future inflation (i.e., expected inflation at time current  $t$  is below actual inflation at  $t$ ). Conversely, a positive memory factor ( $\mathbf{b}$ ) would add to expectations of future inflation above actual current levels.

Our analytical model reveals that, when the inflation rate increases, the price of stocks initially decreases for a short time but thereafter steadily increases. The key to understanding this effect of market participants' memory on stock prices is contained in equation (9). In that equation we assume no memory in the interest rate  $r$ , memory with factor  $\mu$  in stock prices, and memory with factor  $\beta$  in inflation. Importantly, the two factors  $\beta$  and  $\mu$  are not equal because they refer to two different types of memory. To model the observed changes of stock prices we chose the memory factor of stock prices to be stronger than that of inflation (i.e.,  $\mu > \beta$  and  $(\beta - \mu)/\mu < -1$ ). In the example made in the text with  $\beta < 0$ ,  $\mu > 0$  and  $\beta = -1.1\mu$ , equation (9) at time  $t_0$  is negative, since  $\mathbf{j}((t - t_0) / \mathbf{m}^{1/n})$  is unity at  $t = t_0$  and monotonically decreasing to zero, which implies that equation (9) with increasing time will increase (i.e., it will increase to  $C\gamma$ ). It is the greater memory factor of stock prices that causes the decrease and then increase of stock prices. If the memory factor of inflation were stronger than that of stock prices, we would not have the phenomenon observed.

Since our analytical results are consistent with empirical evidence, which has documented that stock returns respond negatively to an increase in inflation in the short run but positively in the long run, they provide new insights into stock price behavior in response to inflation shocks. Namely, our results suggest that stock price memory is stronger than inflation (or price level) memory among market participants. Also, based on our results, we conclude that the Fisher equation is more general than previously believed upon introducing memory into its parameters, as both negative short-run *and* positive long-run effects of inflation shocks on stock prices are possible.

We cannot rule out the possibility that there may be other memory models, or combinations of parameter values in our model, that coincide with stock market evidence concerning the Fisher equation. In this respect the present model is a relatively simple formulation of memory functions that can be readily adapted to more specific patterns of stock price responses to inflation. Future empirical research using vector autoregression methods (VAR) is recommended to explore the exact time path of stock price movements in response to inflation shocks. Impulse response functions generated by VAR models can be employed to show the forecasted responses of stock prices to unexpected movements in inflation rates over time. Also, extension of the present analyses to bonds and real estate prices would be interesting in further work. In general, further research on the time path response of asset prices to

inflation shocks is needed. Do all asset prices have initial negative responses to inflation shocks? When do positive stock price responses to inflation shocks begin to occur and how long do they last? What economic explanations are there for different capital asset responses to inflation shocks?

Finally, it is likely that many other problems in the fields of finance and economics would benefit from introducing memory functions. In this way the diminishing effects of time on variables (e.g., diminishing returns and costs) can be examined in a dynamic framework. As an example, recent work by Caputo (1998) has introduced memory functions in the Hicks equations to show how memory can be employed to model the Lucas phenomenon. Also, as mentioned earlier, analyses of long-run stock price momentum in behavioral finance could benefit from the application of memory functions.

## APPENDIX A

Here we recollect some formulae of  $LT^{-1}$  found in Caputo (1984, 1998), which are used in this paper to derive the formulae from equations (5') through (9). To begin, we define

$$LT^{-1}(u^n / u(\mathbf{I} + \mathbf{m}t^n)) = \quad (A1)$$

$$\begin{aligned} &= \varphi_2(\lambda, \mu, t) = (1/\mu)(\sin \pi n / \pi n) \int_0^\infty \exp(-((\lambda/\mu)u)^{1/n} t) du / (u^2 + 2u \cos \pi n + 1) = \\ &= (1/\mu)\varphi(t/(\mu/\lambda)^{1/n}) \end{aligned}$$

which implies the definition of  $\mathbf{j}(t/(\mathbf{m}/\mathbf{I})^{1/n})$  is a function of only time. Time is measured in units of  $(\mathbf{m}/\mathbf{I})^{1/n}$ , which is analogous to an exponential decay function and commonly referred to as *relaxation time*. The function  $\mathbf{j}(t/(\mathbf{m}/\mathbf{I})^{1/n})$  is decreasing from 1 (at  $t = 0$ ) to zero at  $(t = \infty)$ , i.e.,

$\mathbf{j}_2(\mathbf{I}, \mathbf{m}, 0) = 1/\mathbf{m}$  and  $\mathbf{j}_2(\mathbf{I}, \mathbf{m}, \infty) = 0$ , respectively. It is important to note that the term  $\mathbf{j}_2(\mathbf{I}, \mathbf{m}, t)$  is central to the derivation of the other equations used in this paper. Since

$(1/u(\lambda + \mu u^n)) = (1/\lambda)(1/u - \mu u^n / u(\lambda + \mu u^n))$ , we find that:

$$\varphi_3(\lambda + \mu, t) = LT^{-1}(1/u(\lambda + \mu u^n)) = (1/\lambda) [ \lambda - \mu \varphi_2(\mu, t) ] \quad (A2)$$

$$\varphi_3(\lambda + \mu, 0) = 0 \quad \text{and} \quad (A3)$$

$$\varphi_3(\lambda + \mu, \infty) = 1/\mathbf{I}.$$

We also need

$$\mathbf{j}_4(\mathbf{I}, \mathbf{m}, t) = 1 * \mathbf{j}_2(\mathbf{I}, \mathbf{m}, t) = \quad (A4)$$

$$= (1/\mathbf{m})(\sin \pi n / \pi n) \int_0^\infty [ 1 - \exp(-((\mathbf{I}/\mathbf{m})u)^{1/n} t) ] du / ((\mathbf{I}/\mathbf{m})u)^{1/n} (u^2 + 2u \cos \pi n + 1).$$

It can be verified by differentiation with respect to  $t$  that equation (A4) represents a monotonic increasing function.

The filter applied to  $r$  and to  $\mathbf{p}$  in equation (5'') is based on equation (A1) and is obtained by substituting  $u$  with  $i\mathbf{w}$  in equation (A2). In this way we find that the amplitude spectrum of the filter applied to  $r$  and  $\mathbf{p}$  is

$$1 / \left[ \mathbf{w}^{1-n} \left\{ 1 + (\mathbf{n}\mathbf{w}^n)^2 + 2\mathbf{n}\mathbf{w}^n \cos(\mathbf{n}\mathbf{p}/2) \right\}^{1/2} \right] \quad (A5)$$

which is a low pass filter implying a relative transfer of energy from the high to the low frequencies. The same conclusion is reached for the filters applied to the other terms of equation (A2). This low pass filtering smoothes the functions in the time domain and, according to the theory of fractals, stabilises the system.

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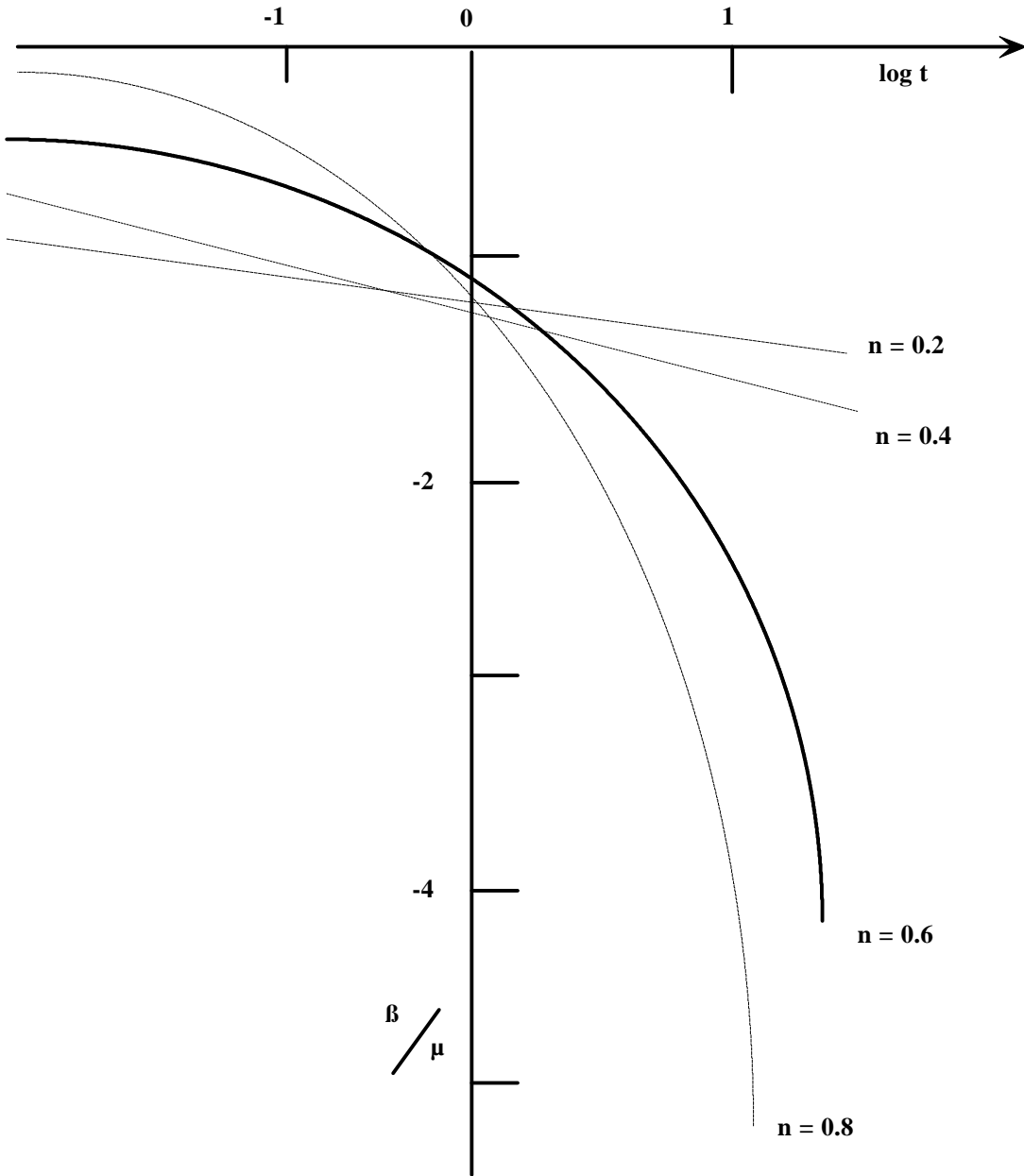
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**Figure 1.** Duration of the negative response of the price of stocks to a positive impulse of inflation, where  $n$  is the fractional order of differentiation. The duration is measured in units of relaxation time  $\mu^{1/n}$ , and  $\beta$  and  $\mu$  are the memory parameters of inflation and of the price of stocks, respectively.