An Enquiry into the Foundations of Economics and Finance

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ABSTRACT

The standard function for the utility of wealth makes no distinction for a *reduction* in the quantity of a good owned irrespective of what this reduction represents e. g. whether it reflects owner's consumption or loss to another party; as if individuals are indifferent between these two actions. This paper suggests that this is a *fundamental* cognitive oversight. It proves formally that, given scarce resources, this error makes the standard definition of utility of wealth and hence expected utility theory invalid and cost free arbitrage impossible. Specifically, this paper disproves subjective probability theory, capital asset pricing model, and the dividend irrelevance theorem. It also identifies the cause of the inconsistency in Walrasian and overlapping-generations general equilibrium models, and highlights the contradiction in other general equilibrium theories. The foundations of a new paradigm are presented, which resolve some of the most stubborn puzzles of the existing literature e.g. corporate control premium.

Keywords: Arbitrage, equilibrium, utility theory, and corporate control premium

1. Introduction

History bears witness that the human mind, both at the individual and mass level, is capable of long-lasting cognitive errors. This paper is about such an error, whilst notwithstanding the contribution of earlier and existing paradigms to the evolution of views, it brings to a head the clash between the standard paradigm in economics and finance vis-à-vis the theoretical, empirical and experimental puzzles that this paradigm gives rise to. It discovers hidden contradictions in all the major standard theories arising from a *single conceptual oversight,* which like an unseen knot binds them together. Knowing precisely what is wrong with the standard paradigm helps revising it properly.

Let us draw a distinction between a *liability* i.e. the obligation to give a good to another party and a *negative asset* represented by the owner's requirement to consume or use up the same good, a distinction that does not arise in a one-man economy. In the history of economic thought, to maintain the individual's level of wealth, Aristotle's (1934, p.273-289) views on reciprocity in a "just" exchange requires identical compensation for the same reduction of wealth represented in the form of either a liability or a negative asset. Subsequently, in all existing "rational" theories of value, Aristotelian reciprocity has led to the identity of the selling price of a good with its buying price at the same date for the same trader in the absence of "friction", hence the "law" of one price. Moreover, hereto literature e.g. Boulding (1987) has not made a distinction in evaluating a liability vis-à-vis a negative asset from the perspective of the owner of the good, or indeed anybody else.

In fact, all formal models in mathematical economics and finance translate the two entirely different concepts of a liability and a negative asset into the same mathematical notation for the same trader. For instance, both ordinal (Wold, 1943-44) and cardinal utility theories (Neumann & Morgenstern, 1947) build on the identity of a liability with a negative asset, and even alternative "rational" theories e. g. regret theory (Loomes & Sugden, 1982) do the same. However, the identity of a liability with a negative asset, which originates from Aristotelian reciprocity, contains hidden contradictions, for it inadvertently identifies the loss of a good to another party with its consumption by its owner. It thus assumes that all individuals are indifferent between losing a good and consuming it, hence the *fundamental* contradiction of existing utility theory. This paper proves that Aristotelian reciprocity is the source of the paradoxes in finance, utility theory and general equilibrium theory.

In particular, it shows that this is the cause of the well-known inconsistency in Walras's general equilibrium model (Garegnani, 1990, p.4), and in Samuelson's (1958) overlapping-generations general equilibrium model. Moreover, it indicates that any other model under this assumption, which leads to the "law" of one price in a "frictionless" market, also collapses. Specifically, it disproves capital asset pricing model, dividend irrelevance theorem and subjective probability theory. It also presents the foundation of a new paradigm that resolves many unresolved puzzles. The structure of the paper is as follows:

Section 2 starts by disproving free arbitrage theory in Subsection 1; Subsections 2.2 to 2.9 unearth hidden contradictions in some of the major theorems of the standard paradigm as *distinctly separate propositions* of this paper. Section 3 presents a fundamental theorem of pricing with important corollaries and lemmas. This theorem introduces *a firmer economic foundation*, which seeks to remove many ambiguities from the existing literature. Section 4 illustrates how the new economic foundation helps resolve one of the most stubborn puzzles in finance i.e. that of corporate control premium. Section 5 concludes the paper.

2. Questioning Aristotelian Reciprocity

Aristotle thought that money was “barren” and hence it would be “unjust” to charge interest on a loan, he also thought it “unfair” to buy a good at one price and sell it at a higher price in the same instance. Economists have long abandoned the first idea, but the second view still predominates on “rational” grounds. This Section examines the validity of the logic of the “law” of one price in a “frictionless” market, and its implications.

2.1. Free Arbitrage Theory

This Subsection proves that when the selling price of a scarce good exceeds its buying price at the same date by a positive infinitesimal amount; this cannot generate a “money pump” i.e. an infinite amount of money. On the contrary, the “law” of one price in a “frictionless” market leads to a “money pump” in an infinite horizon.

There is some realisation (Omberg, 1992, p. 63-64) in finance literature of the availability of free lunches in "frictionless" markets. It is also known that all the major theories of finance rely on free arbitrage theory, and all exhibit *systematic* errors. Moreover, the "instant endowment effect" of Kahneman, Knetsch and Thaler (1990) in experimental economics reveals that an individual assigns a higher selling price than buying price to the same scarce good. This Subsection sheds light on the rationale for these findings.

The standard paradigm assumes that the buying *and* the selling price of a good at any date must be the same in a “frictionless” market for the same trader; otherwise it gives rise to an arbitrage opportunity. Dybvig and Ross (1987) define an arbitrage opportunity as follows:

> "An arbitrage opportunity is an investment strategy that guarantees a positive payoff in some contingency with no possibility of a negative payoff and no net investment. By *assumption,* it is possible to run the arbitrage possibility at arbitrary scale, in other words, an arbitrage opportunity represents a money pump." (Emphasis Added)

It is pointed out here that the latter assumption in the foregoing quotation cannot hold for trading in scarce goods in any instance, hence the fallacy of the free arbitrage theory.

Let us assume that there are a finite number of scarce goods at any date in the economy, and their prices are quoted per units of non-infinitesimal quantities. If the selling price of a good exceeds its buying price for the *same* trader by an infinitesimal amount $\partial p > 0$ at the same date, it requires an infinite quantity of it $q = \infty$ to generate any non-negligible positive payoff $(\partial p)q$ for an arbitrageur. However, any good with an infinite supply at a date is zero-priced, making generation of a"money pump" for scarce goods impossible.

Under the "law" of one price for every scarce good at each date, Samuelson's (1958) overlapping-generations model of general equilibrium leads to a significant "free lunch" for the first generation. This is whilst other generations which receive *and* give the same scarce goods at the *same price* make zero economic profit or loss. This process can be repeated endlessly in this model, leading to a "free lunch" of an infinite size or a "money pump" for the first generation. Moreover, starting from the second generation onwards a similar "money pump" can be generated for the second generation, and likewise for the third generation, etc…. In other words, the model violates the *fundamental* assumption of scarcity of resources, as does free arbitrage theory. If one drops the "law" of one price, which can be logically done as noted earlier, this "money pump" disappears. *QED*

2.2. Fundamental Antinomy of Standard Paradigm

This part of the paper notes that the standard "frictionless" market paradigm overlooks the fact that trade is a time-consuming action in its own right. This gives rise to a fundamental antinomy in the standard paradigm, when time has a positive price or when traders are not indifferent towards risk.

For the same individual, the ownership of an object can start at a *date or* cease at a *date*, taking no time. However, one cannot start *and* cease the ownership of the same object at the same date, as that will be no ownership at all. Thus, ownership must take at least a *positive infinitesimal* amount of time i.e. an *instance* to become *valid* to its owner or anyone else. It is only after the ownership of a good has become valid that its owner can use it. The cash representing the selling price of a good sold takes an instance to become the property of the seller, although the buyer ceases to be its owner at the starting date of this instance. Hence, this cash is *capital tied-up in transit* and is unused and uninvested elsewhere in that instance. In the standard paradigm, the investment return e.g. the interest income lost by a seller on the *capital tied-up in transit* during the instance the transaction becomes valid is ignored. This is true for *all* instances of continuous trading *i.e. for any period, as if either trade takes no time or the rate of interest is zero permanently*.

In particular, the standard paradigm presumes that even an infinite number of actions in the form of consecutive buying and selling of a good take no time, hence there is no tying-up of capital in transit, and no cost of capital to consider for this trade.

The failure to recognise the time taken by the action of trade, and the time value of money in the case of the capital tied-up in transit is a permanent feature of the standard paradigm. That is it holds for all the time in continuous trades, e.g. when repeating endlessly the consecutive actions of buying and selling a good over time. It therefore leads to the presumption of a zero rate of interest for *all time* implicitly, in contrast to the explicit usual assumption of a positive rate of interest in the standard paradigm.

Likewise, the *action* of borrowing, like buying, is assumed to take no time and require no capital tied-up in transit in the instance the transaction becomes valid. In fact, a loan takes an instance to be useable by its borrower, during when this money cannot be used for any other purpose than the validation of this transaction. However the interest expense for the borrower during this instance when the transaction is becoming valid is implicitly ignored.

To see the latter point more clearly, let us note that, as a *permanent* assumption, unrestricted short selling in the standard paradigm means *free* use of the cash proceeds of the sale of the borrowed asset in any instance in which this process takes place, on a continuous basis indefinitely. That is, this cash is presumed to be a free good for an instance, an instance that can be repeated continuously infinitely many times. *Given that any period is a continuum of instances, this leads to the presumption of a zero rate of interest for any period unwittingly, in contrast to the explicit usual assumption of a positive rate of interest in the standard paradigm.*

Similarly, the buyer of a stock with volatile prices must own it for at least an instance for her purchase to be valid, and thus carries the possibility of a loss in this instance, towards which she cannot be invariably indifferent in a world of scarce resources.

However, the standard paradigm denies this, as it does not admit the need for the existence of an instance-long minimum ownership period for a purchased stock. This is an implicit *permanent* assumption for *all instances* of trading in the standard paradigm; thus in *continuous* trading it cumulatively leads to an implicit lack of recognition of risk. This is in contrast to the explicit recognition of risk in many of its models, hence a contradiction.

This antinomy is the root cause of the emergence of the other contradictions studied later here in the "frictionless" market of the standard paradigm, irrespective of whether this market admits risk or not. *QED*

2.3. Capital Asset Pricing Model

The following proves that the capital asset pricing model (Sharpe, 1964) leads to contradiction.

For a given period from now, let R_M represent the random return of the capital market portfolio, and let R and $-R$ be the random returns of two securities X and Y respectively with stochastically dependent returns, then: $Cov(R, R_M) = -Cov(-R, R_M)$. Under CAPM, this leads to two different measures of "systematic risk" one for X: $\frac{Cov(R, R_M)}{Var(R_M)} = \beta$ and another for Y: $\frac{Cov(-R, R_M)}{Var(R_M)} = -\beta$ where $Var(\cdot)$ and $Cov(\cdot)$ are respectively the variance and covariance operators. It also provides two different "expected rates of return": $E(R) = R_F + \beta[E(R_M) - R_F)]$ for X, and $E(-R) = R_F - \beta[E(R_M) - R_F]$ for Y, where $E(\cdot)$ is the expectation operator and R_F is the projected risk free rate. However, the returns of these two securities are stochastically dependent thus: $E(R) \equiv -E(-R)$ hence a contradiction, unless $R_F \equiv 0$. That is to say the standard paradigm implicitly presumes that the time value of money is always zero in contrast to its explicit assumption to the contrary. *QED*

2.4. Corporate Dividend Policy

It is proved here that the standard paradigm holds contradictory positions on the relevance of the dividend policy of a firm to its valuation in a "perfect and complete market with information symmetry".

To see this antinomy, consider a simple case where the firm is all-equity financed and has no foreseeable need to raise new capital. On the one hand, free arbitrage theory famously leads Miller & Modigliani (1961) to the view that dividend policy is irrelevant. On the other, the dividend valuation models presume that any two listed all-equity companies, which have exactly the same risk and stream of dividends, have precisely the same price. That is irrespective of the fact that one company can have a greater amount of retention than the other at all times, for instance one company's retention being twice the other perennially. This makes the valuation of a company very sensitive to its retention or dividend policy, hence a contradiction. *QED*

2.5. Utility of Wealth

This Subsection reveals the antinomies implied by the standard definition of the utility of wealth.

Any argument for justifying free arbitrage theory based on the standard price theory has to refer to the existing definition of the utility of wealth. However, as pointed out in the abstract of this paper, this definition is formulated in a way that it encapsulates the identity of a liability (arising from the loss of a good to another party) with a negative asset (represented by the consumption of the same good by the owner). The latter identity leads to the same compensation against any of these reductions of wealth. This in turn yields the same trader the equality of the selling *and* buying price for the same good concurrently.

It is of course true that both a liability and a negative asset lead to the loss of the same underlying good. However, in the case of a liability whilst the owner loses the good, she is not the beneficiary of this reduction of wealth; whilst in the case of a negative asset, the beneficiary of this reduction of wealth is the owner herself, who consumes the good. This distinction is not picked up in the existing definition of the utility of wealth.

Standard theory offers *two* versions of the function of the utility of wealth, as though wealth could be in two different forms concurrently. In one, wealth is in terms of the quantities of goods owned and owed: hereafter this is referred to as the utility of non-monetary wealth and formulated in small letters u. In the other, wealth is in terms of the cash price of these goods, although the inefficiency of keeping all wealth in cash form is acknowledged in that paradigm.

Hereafter, the latter is called the utility of monetary wealth (i.e. the individual's equity capital) and is denoted in capital letters U. In standard theory, the utility of monetary wealth is "derived" as the indirect utility function when prices are constant, and hence the reason for dropping the price argument in that function. This *appears* formally valid for the special case when the *net* quantity of each good i.e. the quantity owned less the quantity owed is non-negative. However, this *netting* leads to contradictions, as explained here:

Given an optimum bundle of assets and liabilities in non-monetary terms, the utility of monetary wealth becomes the utility of the cash price of her property rights *net* of the cash price of her obligations. Thus, the existing definition of the utility of monetary wealth W

represented by $U(W)$ *presupposes* a matched portfolio of receivable $X > 0$ and payable monies $-X$ arising from buying and selling goods *never* changes the individual's welfare:

$$U(W) \equiv U(W + X - X).$$

Thus, the function defining the utility of monetary wealth presumes that all individuals are indifferent over doing no trade as against doing any trade that nets to zero in monetary terms, despite the fact that this function is already an optimised utility function, hence a contradiction.

To see this fallacy more clearly, one may note the following. Under standard theory let the individual's optimum non-monetary wealth comprise a convex combination of x and y as two distinct complementary bundles of positive quantities of goods with the same price W per each bundle, and where x is strictly preferred to y. If the utility function u over x and y is strictly quasiconcave and the individual's wealth W is $tx + (1-t)y$ where $0 < t < 1$ then:

$$u[tx + (1-t)y] > \min[u(x), u(y)] = u(y).$$

Thus, by selling all x in this combination and with its cash proceeds of $X = tW$ buying y, hence replacing x with y whilst keeping unchanged the monetary value of the individual's wealth at $W - X + X \equiv W$, the individual deteriorates his welfare. *That is to say doing nothing is preferable to such a trade, although this trade nets to zero in monetary terms.*

For example if x and y *are two distinct complementary goods with the same marginal utility and hence the same price, the same result follows,* as long as x is strictly preferred to y. If the utility function were strictly quasiconvex, a similar contradiction would arise. The only case where no change of welfare arises is when all *preferences are linear*. Otherwise, one can always find such x and y in a strictly quasiconcave (or strictly quasiconvex) bounded interval that leads to a change in welfare, hence a contradiction. *QED*

2.6. Utility of Consumption

This Subsection points out the contradictions implied by the utility of consumption.

In the standard paradigm, when an individual is said to prefer one good to another, it is *implicitly* assumed that this preference is in terms of both ownership *and* consumption of these goods concurrently, although these are two entirely different concepts. That is to say the meaning of the statement: "Sally prefers apples to pears" is "Sally prefers to *own* apples to pears" *and* "Sally prefers to *consume* apples to pears" concurrently. However, as a possible means of avoiding the contradictions in Subsection 2.5, and given the fact that preferences are often defined over consumption bundles in the standard paradigm, it is possible to argue that preferences should be considered defined strictly in terms of consumption *alone*. That is to say the statement: "Sally prefers apples to pears" should be taken to mean "Sally prefers to *consume* apples to pears" and *not* "Sally prefers to *own* apples to pears". Under this view, preferences do *not* depend on the individual's ownership or non-ownership of the consumption bundles.

This view means that the welfare function defined by these preference relations measures utility of *consumption*, and not utility of *wealth,* whereby utility of consumption is independent of wealth. Clearly, utility of consumption is unaffected by the amount of any liability. However, this reinterpretation of the standard paradigm does not remove this underlying deep-seated contradiction, as it implies that the individual is indifferent towards consuming a good, irrespective of whether she owns it (as a negative asset) or owes it (as a liability). Hence, the budget constraint and scarcity of resources have no impact on the individual's welfare. For, the individual's welfare, indicated by her utility of consumption, can be improved by consuming ever-increasing *borrowed* goods. This view, which implies that the budget constraint is no barrier for improving the individual's welfare, leads to the collapse of consumer price theory. *QED*

2.7. Expected Utility Theory

This part of the paper explains the theoretical reason behind the breakdown of expected utility theory in experience and in experiments.

One finds classic examples of these breakdowns in Friedman and Savage (1948), Allais (1953), Markowitz (1995, p.219), Elsberg (1961), Samuelson (1963) and experiments of Tversky and Kahneman (1986). Duncan (1977) proves by a formal example that for multivariate random monetary wealth there is no *unique* certainty equivalent. However, it escapes him to note that given the fact that the utility function is one-to-one, therefore the certainty equivalent *must* be unique, hence a contradiction.

The following proofs are against expected utility theory. Proof 2 can be seen as a generalisation of the example of Duncan (1977).

Proof 1:

In the standard paradigm, the individual can have preferences not only on sure outcomes, but also on probability distributions. (Arrow (2001) points out that this is one of Harsanyi's contributions to the standard paradigm.) Let us assume, for simplicity, the individual has no wealth but can borrow, and considers playing gamble G_X where X represents the random monetary payoff of this gamble. Under expected utility theory, the individual can write for $\underset{X}{G}$ (Ingersoll, 1987, p.33):

$$V[f(X)] \equiv \int U(X)dF(X).$$

Following Ingersoll (1987), V and U must represent the individual's ordinal and cardinal utility functions respectively, where f is the probability density function and F is the cumulative probability function.

As an *alternative*, the individual considers playing gamble $\underset{-X}{G}$, where X and $-X$ are stochastically dependent monies. The same individual can thus write for $\underset{-X}{G}$:

$$V[f(-X)] \equiv \int U(-X)dF(-X).$$

As $f(X) \equiv f(-X)$ and $F(X) \equiv F(-X)$, a comparison of the foregoing identities leads to $U(X) \equiv U(-X)$, which is of course impossible, unless $X \equiv 0$.

It can be argued that if $\underset{-X}{G}$ is taken as $-\underset{X}{G}$, i.e. if one assumes that for this individual *offering* gamble $\underset{X}{G}$ to another player is the same as *playing* gamble $\underset{-X}{G}$ i.e. $-\underset{X}{G} \sim \underset{-X}{G}$, where $\sim$ denotes indifference, then this contradiction does not arise. However, for the same individual, given a strictly concave and strictly monotonic bounded utility function U, following the Jensen inequality, when she considers playing gamble $\underset{X}{G}$, one can write:

$E\left[U(X)\right] \equiv U\left[E(X) - {}_{X}h\right]$ where ${}_{X}h > 0$; and if she considers playing gamble $\underset{-X}{G}$ then: $E\left[U(-X)\right] \equiv U\left[E(-X) - {}_{-X}h\right]$ where ${}_{-X}h > 0$.

Therefore: $\underset{X}{G} \square [E(X) - {}_X h]$ and $\underset{-X}{G} \square [E(-X) - {}_{-X} h]$. Thus, if one assumes $-\underset{X}{G} \square \underset{-X}{G}$, then: $-[E(X) - {}_X h] \square [E(-X) - {}_{-X} h]$. Hence, ${}_X h \square -{}_{-X} h$ which is impossible as ${}_X h$ and ${}_{-X} h$ are both positive, unless $X \equiv 0$. *QED*

Proof 2:

For an individual with total wealth $X + Y$ and a strictly concave and strictly monotonic bounded utility function U, where X and Y are both non-degenerate random monetary amounts, in the absence of the expected utility theory, one can define ${}_{X+Y} h$ such that:

(2.1) $$E[U(X+Y)] = E\{U[E(X+Y) - {}_{X+Y} h]\} = U[E(X) + E(Y) - {}_{X+Y} h\ ]$$ with $E(\square)$ being the expectation operator. Following the Jensen inequality, when $X + Y$ is certain, then ${}_{X+Y} h = 0$, otherwise ${}_{X+Y} h > 0$.

Thus $E(X+Y) - {}_{X+Y} h$ is defined as the expected utility equaliser of $X + Y$. If this individual had only the sure wealth W and non-degenerate random X with mean $E(X)$, one can define ${}_X h_W$ such that

(2.2) $$E\big[U(W+X)\big] = U\big[E(W+X) - {}_X h_W\big] = U\big[W + E(X) - {}_X h_W\big],$$ where ${}_X h_W > 0$.

Thus $E(X) - {}_X h_W$ can be defined as the expected utility equaliser of X given prior sure wealth W. For the individual with total wealth $X + Y$, where X and Y are random, irrespective of whether $X + Y$ is certain or not, let us define the expected utility equaliser of X given prior uncertain wealth Y to be $E(X) - {}_X h_Y$ so that

(2.3) $$E[U(X+Y)] = E\{U[E(X) - {}_X h_Y + Y]\}.$$

It follows that the expected utility equaliser of Y given prior wealth X is

$E(Y)-{}_Y h_X$ where

(2.4) $$E[U(X+Y)] = E\{U[E(Y)-{}_Y h_X + X]\}.$$

The expected utility equaliser as defined here may remind one of Pratt's (1964) measure of certainty equivalent and Machina's (1982) measure of conditional certainty equivalent; but it is a wider concept, as it does not rely on the validity of the expected utility theory.

Under expected utility theory, one is indifferent towards replacing an element of wealth as long as the same level of expected utility for the whole of wealth is retained. Therefore, according to this theory, one can replace X and Y by their expected utility equalisers, and vice versa, under all circumstances.

If ${}_Y h_X$ is derived in (2.4) then $W' = E(Y)-{}_Y h_X$ becomes certain. Following (2.2), one must have ${}_X h_{W'} > 0$. Substituting Y in place of its expected utility equaliser $W' = E(Y)-{}_Y h_X$ in ${}_X h_{W'} > 0$, one obtains ${}_X h_Y$ >0.

Similarly, if ${}_X h_Y$ is ascertained in (2.3) then $W'' = E(X)-{}_X h_Y$ becomes certain. Following (2.2), one must have ${}_Y h_{W''} > 0$. Substituting X in place of its expected utility equaliser $W'' = E(X)-{}_X h_Y$ in ${}_Y h_{W''} > 0$, one obtains ${}_Y h_X$ >0.

Hence invariably ${}_X h_Y$ >0 and ${}_Y h_X$ >0. It is worth noting here again that both Pratt's certainty equivalent and Machina's conditional certainty equivalent measures are lower than the corresponding means for individuals with strictly concave and strictly monotonic bounded utility functions, consistent with these findings. Replacing X and Y by their expected utility equalisers one obtains

(2.5) $$E[U(X+Y)] = E\{U[E(X) - {}_X h_Y + E(Y) - {}_Y h_X]\}.$$

Thus

(2.6) $$E[U(X+Y)] = U[E(X) - {}_X h_Y + E(Y) - {}_Y h_X].$$

However, from a comparison of equalities (2.1) and (2.6), and given the one-to-one U :

(2.7) $${}_{X+Y}h = {}_X h_Y + {}_Y h_X .$$

This is impossible. The reason is for when $X+Y$ is certain: ${}_X h_Y > 0$ and ${}_Y h_X > 0$ whilst ${}_{X+Y}h = 0$. That is expected utility theory cannot hold for multivariate random wealth. In fact, for (2.7) and by implication for expected utility theory to hold for *any* X and Y, then ${}_X h_Y = {}_Y h_X = {}_{X+Y}h = 0$ i.e. preferences should be linear. Otherwise, there are such random X and Y in a strictly concave (or strictly convex) bounded interval that can lead to this contradiction. *QED*

2.9. Link between Expected Utility Theory and De Finetti's Exchangeability And the Failure of Subjective Probability Theory

This Subsection proves that the implications of expected utility theory and De Finetti's exchangeability are the same, and hence the reason for the collapse of subjective probability theory.

Expected utility hypothesis has exercised the minds of some of the greatest mathematicians over centuries e.g. the Bernoulli's, Cramer, Laplace, Menger, Neumann, etc. It is thus worthwhile to reflect on what escaped their attention in their conceptualisation, so as such a pitfall is avoided in the future. To do so let us be clear about the meaning of the concepts and the underlying assumptions of this theory:

Definitions:

A *process* can be conceived as a set of actions that are somehow linked by a common property. If the durations of two actions coincide only on one date (of zero duration) at most, the actions are said to be *non-overlapping*. The *order of realisation over time* of the different actions in a given process comprising a non-overlapping set of actions can be either *sure* viz. the forecast order is available and certain to be the same as the actual for the individual, or not so i.e. it is *unsure.* If the order in which any of two alternative actions of a process are realised does not matter to the individual, she is said to be i*mpartial* towards them. This can be the case if this order is irrelevant to her, or the individual has no preference over this order, or is indifferent towards this order.

Proof:

If each action in *a non-overlapping* set of actions of a process has an *unsure order of realisation*, the individual cannot be assumed to be invariably *impartial* towards the order of the realisation of these actions, when each of these actions has a different impact on her welfare. However, expected utility theory assumes impartiality in this respect, as *it assumes that the order of the realisation of the different prizes of a lottery draw does not matter* to the gambler or anybody else. This is inconsistent with scarcity of resources, and breaks down the expected utility hypothesis. The point overlooked here is that the fact that the lottery player does not have a free *choice* over this order does not mean that she is *impartial* towards this order.

This impartiality is also explicit in De Finetti's (1980) exchangeability (i.e. *indifference to the* *permutations of the same objects in a combination*), which for the same reason cannot hold over scarce private goods.

In fact, it is perfectly *reasonable to prefer* to consume one good *prior* to another in terms of the *timing* of these actions. Moreover, the standard paradigm acknowledges that an individual prefers to have a greater amount of money sooner than later. However, in expected utility theory, for consistency with the axioms of probability theory, the individual is taken to be *impartial* towards the order in which each stochastic outcome (action) of a random process is realised. This is done, by supposing explicitly that: *the order of the incidence of the lottery prizes is not irrelevant to her.*

It is possible to argue that this *timing* difference is too small to worry about. However, as it can occur infinitely many times, it cannot be negligible in any theoretical work, in Isaac Newton's words: "In mathematics the minutest errors are not to be neglected". Subjective probability theory rests on De Finetti's exchangeability, with or without expected utility theory. Thus, the failure of De Finetti's exchangeability means subjective probability theory cannot be valid. *QED*

2.9. General Equilibrium Models

This Subsection explains the implications of Aristotelian reciprocity for the major neo-classical general equilibrium models.

The neo-classical excess demand, net supply and profit functions (or correspondences) are defined assuming that the buying *and* selling price of the same good is equal from the perspective of every trader (Arrow and Hahn, 1971). That is including a trader who buys and sells the same good, e.g. a farmer who produces the same seeds that she grows, and buys and sells her input and output. In each of the following models a contradiction shows up, as it is assumed that the "law" of one price for the same good for the same trader at each date holds.

Walras (1874-7, p.171) assumed that the *buying* price of all inputs i.e. "cost of production" equals the *selling* price of the relevant output for each producer. However, he later discovered when considering *reproduced* capital goods (e.g. machinery), which producers can have both as input and as output at the same price concurrently, that his model is over-determined, as Eatwell, Milgate and Newman (1990, p. xii) point out and Garegnani (1990, p.4) explains.

Eatwell, Milgate and Newman (1990, p. xiii) note that *under certainty* in a wide range of "intertemporal" models of equilibria, including the Arrow-Debreu:

> "The rate of return on non-reproduced capital goods is lower than the rate of return on reproduced capital goods. "

However, the same models leave open the opportunity for free arbitrage, which requires a uniform rate of return for all risk-free investments in this paradigm, hence their fundamental contradiction, a point which appears overlooked hereto. *QED*

3. Putting Economic Theory on a Firmer Foundation

This Section reviews standard economic theory in the light of Section 2 by first considering a risk free "frictionless" market. It explains that whilst the same trader can buy or sell a good at the same price, quoted at the same date, however, she cannot buy and sell it at the same price without losing money. In fact, as long as her investments can instantaneously earn a positive return, the selling price must exceed the buying price for the same trader at the same date by a positive infinitesimal amount. Moreover, it points out that a perfect hedge is not cost free, and prospectively at a finite number of dates, the selling price can exceed the buying price significantly for goods with volatile prices without leading to a "money pump". This leads to the possibility of the realisation of profits for risk takers.

Theorem:

The "law" of one price at each date for trading in a scarce private good cannot hold for an arbitrageur. In fact, the selling price of a scarce private good must exceed its buying price by a positive infinitesimal amount at the same date for a trader who engages in both these actions in the same instance in a risk free "frictionless" market with a positive rate of investment return. This is necessary so as, over a non-negligible period, the trader does not suffer a non-negligible increasing loss and thus eventually become bankrupt, or, benefit from a "money pump".

Assumptions, definitions and their justifications: ***a new foundation***

Debreu (1987) notes:

> "The very definition of an economic concept is usually marred by a substantial margin of ambiguity".

The following definitions and assumptions delineate the boundaries amongst some of these concepts so that they will not clash over the same territory in their "margins of ambiguity", causing a contradiction. They thus draw out the distinction between different ideas that *appear* alike in standard economic theory, and which underlie the contradictions brought to light later in the paper.

Readers are assumed to have the same understanding of the primary concepts describing time, objects, and actions. The concept of the *infinitesimal* features frequently in this theorem, and although it is at least over 300 years old, its precise characterisation is provided comparatively recently, such as that in non-standard analysis (Robinson, 1972) by "extending" the real numbers. The absolute value of any infinitesimal is smaller than all positive real numbers in this "extended" field, in which this theorem holds and where a number that is not zero is called *non-negligible* for the purposes of this theorem, and a non-negligible number that is not infinitesimal is described as *significant.*

Let time be represented by the set of all extended non-negative real numbers along an "extended" real line. A point on this line is referred to as a *date*. For the two dates t_i and t_j where $t_j > t_i$ the *duration* between the date at the beginning t_i and the end date t_j is $\Delta t = t_j - t_i$. An *instance* T in this paper is a *positive infinitesimal* duration such that $\Delta t = T$. Two *consecutive* durations in continuous time are defined as having different end *dates* with the end *date* of the first duration being the beginning of the next.

An action in its simplest form, other than one that indicates the start and end of an action, takes at least an instance to materialise. Actions, which pairwise occur in *consecutive durations* in *continuous-time,* are defined as *consecutive actions*.

The exercise of the right of private ownership as an action can be defined as follows. An individual is said to *own* a positive quantity of an object, if she can have immediate, direct and exclusive *access* to it for any purpose offered by that object throughout her ownership duration. An individual can start *or* cease ownership of an object at a date, but she cannot start *and* cease the ownership of the same object at the same date, as that represents no ownership at all. Thus the ownership duration starts at a date and ceases at a date, and *must* at least last an instance to become *valid.* That is to say, to own an object one must have the right of *access* to it for at least an instance and not merely for a date, as a date has no duration, and cannot represent evidence of ownership even to the owner.

Clearly, one cannot say ownership starts in any instance before the starting date of access to the object concerned. Similarly, if there is an end to ownership, this end can only be represented by a date when this access ceases and not any time prior to this date, as the same owner has access to the object during *the entire* instance prior to this date. *Thus, whilst it takes at least an instance to own an object, once the object is owned its ownership ceases at a date, if it ceases at all.* The standard paradigm overlooks the need for the existence of an instance as a minimum duration for the ownership of an object to be realised i.e. become *valid*.

It follows that actions such as buying, selling and gambling must take at least an instance. The reason is they lead to a change of ownership over objects, and ownership takes at least an instance to materialise.

One may argue that an action such as buying, selling and gambling can be conceived to occur at a *date*, taking no time as in "static" or "discrete-time" models. However, under this conception, it is impossible to differentiate between two actions irrespective of whether they are *consecutive* or *non-consecutive* actions. Economic theory has used "static" or "discrete-time" modelling as "approximations" to continuous-time modelling, without any analytical proof that such "approximations" do not lead to serious errors. *That is as if* *even an infinite number of consecutive trades can never take any significant amount of time.*

This paper illustrates that given that time has often a price in economic theory, these "approximations", despite their appearances, are deeply misleading in the final analysis. The following three paragraphs provide the fundamental reason for this confusion:

Let us leave out actions that can be assumed to take no time, viz. only *starting* or only *stopping* an action, from what are referred to as actions hereafter. Actions in their durations can be completely overlapping i.e. coincide on all dates, partly overlapping i.e. coincide on more than one date (but not on all dates), or non-overlapping i.e. coincide on no more than one date. Let us distinguish between *concurrent* actions which are overlapping in their entire durations, and *sequential* actions i.e. a series of non-overlapping actions which are consecutive in their durations.

Let us suppose there are N actions $a_1, a_2, a_3, ..., a_N$ that can take place *either concurrently* i.e. overlap completely in their timing with the same starting date and the same finishing date in instance ∂t, or can take place *sequentially* i.e. during consecutive instances $T_1, T_2, T_3, ..., T_N$ continuously in that *order*. The duration T, in which all these actions occur is also infinitesimal, as $T = T_1 + T_2 + T_3 + ... + T_N$ when N is finite.

These two sets of actions differ in the order they happen, and are not identical, although they all can take place in the same instance $T \equiv \partial t$, and *appear* "simultaneous". *Thus, in a formal model, one action can precede, coincide with or succeed another even in an instance, if the number of all non-overlapping actions in any instance in that model is finite. In contrast, the number of concurrent actions in any instance need not be finite.* Unwittingly, "static" and "discrete-time" economic models do not make this distinction, as if an infinite number of consecutive actions *always* take only a negligible instance to materialise, they thus lead to confusion. Moreover, existing continuous-time models rely on some of the assumptions of the "static" or "discrete-time" models, and are thus not internally consistent.

A good for an individual in this theorem is defined to be an object that she deems her owning of any *positive* amount of its quantity is preferable than not owning it. That is when these two alternatives are judged concurrently, *irrespective of any other consideration*. It is worth noting that this is merely a test of attitude towards two *hypothetical concurrent alternatives*, and does not require the materialisation of any objective action, and ignores the price of the object and individual's actual circumstances including her existing wealth, consumption or storage capacity, obligations to others, etc. Moreover, it is assumed that the source from which the object is received, and the context, in which the object appears, does not have any impact on this test. Without these abstractions, the same object cannot be regarded as a good for *all* individuals *concurrently*.

In the foregoing definition preferences over ownership of objects are not confused with preferences over their consumption or usage. Standard utility theory tacitly presumes preferences over ownership of objects are concurrently the same as preferences over their consumption (or personal usage). Thus, this definition represents a reduction in the implicit assumptions of the standard paradigm.

In fact, this definition permits an individual to prefer greater ownership but not necessarily greater consumption of the same good concurrently, which is necessary in respect of durable goods and goods with seasonal fluctuations in their production and consumption.

This definition legitimises the description of a financial contract (e.g. a loan) as a good, which is implicitly accepted in the standard paradigm. *However, it should be noted that this theorem is not concerned with relative preferences i.e. how one good might be preferred to another.*

A scarce object is one that is not available in an infinite quantity in any instance, when measured in units of a positive significant size. Individuals are assumed to be unanimous on the identity of all scarce goods. Scarce private goods are excludable and diminishable (as in standard paradigm), and individuals have *exclusive* property rights on them at all times by social convention. This right means that no amount of a scarce private good is given up freely and willingly in any instance without receiving a positive quantity of another scarce good, hence no "free lunches". The reason why existing literature fails to define scarce private goods is because it cannot eliminate "free lunches".

For greater focus on key issues, this theorem ignores "transaction costs", as in the standard "frictionless" abstract paradigm. Thus, the scarce private good considered in this theorem, is an *abstract* object that does not need any storage and does not require labour for its procurement, handling or administration. Moreover, the good does not change in quality or quantity in an instance of trading.

An individual's *wealth* by the end of an instance of the ownership of that wealth comprises a set of scarce private goods that she owns, known as *assets*, and concurrently, a set of scarce private goods that she *owes* i.e. which she must give up *prospectively*, known as *liabilities*. There exits a subtle timing difference, amongst other things, between *a negative asset i.e. the requirement to consume or use up a scarce private good owned* and a liability i.e. the obligation to give up that good to another party, which can now be explained. When the individual has already obtained valid ownership of a specific item of a scarce private good e.g. a gold ring as an asset, she can consume it as a negative asset *and* own it during its consumption concurrently. However, she cannot own it as an asset, *and* lose it to another party as a liability concurrently.

Money as a means of exchange is a *unique* scarce private good by virtue of its timesaving characteristic in reducing the number of exchanges. This becomes particularly clear if one engages in an infinite number of consecutive transactions over a significant duration. It is assumed this unique quality is recognised and unanimously assigned to a single durable and divisible object. The number of transactions exchanging money for other scarce private goods in significant quantities and prices is assumed to be finite in any instance, if this number were infinite; money could not be scarce any more and have a "time value".

The *date of exchange* represents the starting date of ownership for a buyer and the end date of ownership for the seller of the good being exchanged. The prices of all scarce private goods are quoted in money terms, and are assumed to be available for each date, and they are *positive, significant and finite* numbers. The "law" of one price holds if only a single price is *supposed* to exist for each date for each scarce private good that is traded.

The money available in the economy is assumed to be such that the individual can always find opportunities to earn a positive return from her money if invested in any instance. The individual, thus at all times, keeps all her money invested if not used in exchange.

It is impossible for the same individual (say, I_1) to buy from one party and sell to another party (say, I_2) the same specific item (say, with a unique serial number) of a given private good *concurrently*. Otherwise both I_1 and I_2 would be *buying the same item of this good concurrently*. In other words the same item of this good would have two owners, which is impossible by the definition of a scarce private good. Thus, this would be an *invalid* trade.

For a buyer, it is only after the *elapse of at least an instance* of ownership of a good *since the date of exchange* following its purchase that this transaction becomes *valid.* However, it is *at the date of exchange* that she *ceases* the ownership of the money representing its buying price. Similarly, the seller of a scarce private good *ceases* her ownership of it *at the date of exchange*. However, her *ownership* of the money she accepts for it takes at least *an instance since the date of exchange* to become *valid*. Prior to that point during this exchange, this money is *capital tied-up in transit*, for neither the buyer nor the seller can use it for any other purpose than this exchange.

To cease the ownership of a scarce private good one must have owned it for at least an instance already. Thus to sell a good, one must own it in the first place. The only way one may justify *short* selling a good, by way of borrowing and selling it, is to assume that the borrowing contract leads to the *ownership* of the good on the part of the borrower for the duration of the loan, and the lender is compensated accordingly. This view of a borrowing contract is implied by the standard paradigm in finance theory, and it is accepted in this paper for simplicity.

Thus, the lender of scarce private good, say money will *not* be its owner from the starting date of the duration of the loan to its end date. However, she will be the owner of the right of its return with interest by the end date of the loan. On the other hand, borrowing such a good for a period leads not only to the borrower *owning* it in that period, but also for the borrower *owing* the obligation to return it with interest by the end date of the loan.

To borrow a scarce private good, one must have it as a loan for at least an instance otherwise the loan transaction is not *valid.* The lender of any scarce private good, including money, requires compensation from her borrower, by definition. This is true for a loan of any length of time, including an *instance.* This compensation takes the form of a *positive* amount of interest for every instance payable by the end date of a loan by assumption.

Let us assume that the amount of a loan at the start date of a loan is X , and the *discrete* rate of interest payable on loan X over its duration is R_X , such that the accrued interest at the end date of the loan is $R_X X$. Then the total payable amount to the lender at the end date of the loan is $(1+R_X)X$. The *continuous* equivalent rate of R_X is r_X where $1+R_X \equiv e^{r_X}$ and e is Euler's number.

Therefore if the good is borrowed (lent) at price X , reflecting the buying (selling) price of the good at the starting date of the loan, the accrued interest at the end of an instance of his loan is $R_X X$ by definition. The latter is a *positive infinitesimal amount* as the discrete instantaneous rate of interest R_X is also *a positive infinitesimal* number.

Clearly, the rate of interest for the loan of any scarce private good for any finite period is finite. One may presume that the *instantaneous* rate of interest in any instance is so small that it can be ignored i.e. taken as zero. However, as any significant duration is the sum of an infinite number of consecutive instances, this leads to the rate of interest to be taken as zero for any period, which cannot be true for a scarce private good e.g. money.

It is not presupposed, as it is done so implicitly in the standard paradigm that an individual is indifferent between (I) doing nothing and (II) acquiring one item and disposing another item of the same scarce private good at the same price in the same quantity concurrently. The reason is option (II) comprises time-consuming actions, and in a world when time has a price this cannot be justified, particularly when such actions can take place infinitely many times.

The following forecasting rules are admitted: Information symmetry is assumed for all. If the buying price of a scarce private good is B at the present date t_1 for a trader, the present value of any of its future buying prices is also B for the same trader. Likewise, if the selling price of a scarce private good is S at the present date t_1, the present value of *any* of its future selling prices is also S for the same trader. The buying price of any cash flow to its owner is the present value of the cash from each date direct *access* to it starts (represented by a positive amount), and if relevant, ceases (represented by a negative amount). Thus, if there is no likelihood that a prospective sum of cash becomes accessible, it has no present value.

In the standard paradigm "risk" is taken as a prospective possible (i.e. not sure) gain *or* loss with known stochastic characteristics, and if the latter is not available, "uncertainty" is said to prevail. Given homogeneous expectations, under subjective probability theory, the distinction between "risk" and "uncertainty" disappears. This paper brings to light the antinomies based on these conceptualisations in Subsections 2.3, 2.7 and 2.8.

For a new definition of risk, let us first distinguish between losses and voluntary consumption. A *loss* is defined here as a reduction of wealth, a reduction that does not *concurrently* represent a consumption of the owner of that wealth. This Section suggests that *risk to wealth for its owner at the start of any period is* a *possible loss in that period or a possible involuntary increase in consumption needs in that period. Clearly, risk assessment is at least partly a subjective concept.*

A sure parameter in respect of a future date is one, for which there is a unanimous forecast and the forecast and actual out-turns are *certain* to be the same. Clearly, this requires the kind of collective subjectivity in respect of this parameter, which for its believers, is indistinguishable from objectivity. The only requirement for a *risk free* market in this theorem is that the profit from trading is a sure parameter and it is zero. In a *risky* market it is not a pre-requisite to have a profit forecast from trading in a good to engage in its exchange: and forecast profits, if available, are not sure parameters. Given that money is a scarce good, it is assumed that all traders seek to maximise their forecast profits measured in money terms, if available, hence forecast profits from trading cannot be below zero, and if they are above zero, arbitrageurs seek to minimise it. The latter is a *social* result of *individual* profit maximisation in a market with at least three independently acting traders.

An intermediary in its most general sense is one who acquires and disposes of the same scarce private good. Thus, an arbitrageur is an intermediary, whose existence is *necessary* for the price formation process described in this theorem. This avoids introducing artificially the Walrasian auctioneer, and also the difficulties of not introducing it as Arrow and Hahn (1971, p.325) recognise:

> "What is happening now is that, having decided on one idealisation (perfect competition), we run into what must be taken to be *logical difficulties* unless we import another idealisation: the auctioneer. " (Emphasis Added)

Leaving out barter agreements from this study for simplicity, buying and selling any scarce private good requires the individual to acquire and dispose of money. Therefore, it is not only a bank, but also any other trader, who is an intermediary in a non-barter economy, hence the importance of the study of the role of the intermediary, as in this theorem. The "frictionless" market of this theorem is risk free by assumption, however, the impact of *price volatility i.e. the possibility of significant price changes,* is considered in the lemmas.

Proof of the Theorem:

An exchange e.g. buying or selling leads to new owners of scarce private goods. For whom it takes an *instance* for this new ownership to become *valid*, during when these goods cannot be used in anyway e.g. consumption, production, or other trade. On the other hand, the previous owner of each good ceases her ownership of that good at the starting *date* of this instance. This leads to an instantaneous economic loss for each trader, which has to be recovered if the new owners were to sell their newly acquired goods. It is this, which leads to the selling price to exceed the buying price by a positive infinitesimal amount at the same date for the same trader. If this is not possible, no trade takes place.

In an economy with money as its only means of exchange, this phenomenon takes the following form from the perspective of a seller. The seller ceases the ownership of the good she has sold from the starting date of the instance of exchange. However, it takes an instance since that date for the compensation the seller accepts to be owned by her i.e. for the money representing the selling price of the good to become a useable property for her, which leaves this money uninvested elsewhere in that instance.

Similarly, a buyer ceases the ownership of the money representing the buying price of the good from the starting date of the instance of exchange. After this date, it takes her an instance to *own* the good she buys i.e. for the good to become a useable property for her.

Consider a trader, who buys and sells in that order the same quantity of a scarce private good N times continuously in N consecutive rounds of buying and selling, with each round representing two consecutive instances as follows. She starts buying this good at date t_0 and starts selling the same item of the good (e.g. with the same serial number) at date t_1 when its selling price is S, where $T_1 = t_1 - t_0$ represents the first instance of the first round of trade. In the second instance of the first round of trade, she starts buying another item of the same good at date t_1 when its buying price is B, and starts selling it at date t_2. Here $T_2 = t_2 - t_1$ represents the second instance of the first round of trade and the first instance of the second round. Likewise, this process continues for the next consecutive instances of each round for N times.

The trader, with no prior ownership of this good, must buy it *first* in T_1 and again in T_2 in order to trade in it from then on. If the trader chooses to stop trading immediately after t_N, she will end up with the good she bought in the instance starting at t_N, and also the cash proceeds from the last sale she made then. This good and this cash will be in her ownership at t_{N+1}. Thus, at prices ruling at t_{N+1} she recovers *access* to her investment of the good and the cash she put in at t_1; *therefore her investment is accessible to her anytime she chooses.* This investment is her *total capital tied-up in transit* in this trade. In the meantime, the forecast economic profit or loss at t_1 from this process can be computed as follows:

By definition, at t_1 the present value of *any* future buying price of this good is B, and at t_1 the present value of *any* of its future selling prices is S. At t_1 in present value terms, the excess of the sales revenue over the buying cost of the good from all trades since t_1 is the forecast economic profit π where: $\pi = N[S(1+R_S)^{-1} - B]$. This is where $R_S > 0$ a positive infinitesimal number represents the trader's forecast return (in discrete form) from investing S during T_2. This reflects the fact that the money representing the selling price S at date t_1 becomes the seller's property only in the instance after t_1 but by t_2, during when she loses her return from investing this money.

At date t_1, the present value of the buying price of the trader's investment outlay is $2B$, and at the same date the present value of her recovery of this outlay is $2S$, if she stops immediately after t_N for all $N > 1$ and sells her inventory. Thus, at date t_1 the buying price of *her total capital tied-up in transit* during anytime in this trade is $2B$, and its selling price is $2S$.

In a *risk free market* the actual and forecast profits are zero by assumption. Thus, for the economic profit π from this process to be zero, one must have $S = B(1+R_S)$. This is when the selling price S exceeds the buying price B by a positive infinitesimal amount at the same date. t_1. This remains true even when N tends to infinity in a non-negligible period of continuous trading.

Thus, zero-economic profit is a necessary and sufficient condition for the "law" of one price at a date to break down. Hence, the selling price of a scarce private good must exceed its buying price for the same trader at the same date by an infinitesimal positive amount. Moreover, these prices reflect her *instantaneous rate of investment return. The role of arbitrageurs is to ensure that the spread between the buying price and the selling price remains infinitesimal in a risk free market.*

If there is more than one buying price for a scarce private good for the same trader, or there is no unanimous forecast of R_S, there can also be many selling prices for the same scarce private good at each date. However, the difference between any pair of buying and selling prices at the same date for the *same trader* must still be infinitesimal i.e. the highest selling price cannot exceed the lowest buying price at the same date by more than a positive infinitesimal amount, and this retains the risk free nature of this market, as defined in this theorem. *QED*

Further implications of this theorem in a risk free market are considered in the following corollaries. Thereafter, subsequent lemmas consider the implications of the foregoing analysis in a market with price volatility.

Corollary 1:

It is impossible in a risk free "frictionless" market with a positive instantaneous investment return over a non-negligible period of continuous trading to buy a scarce private good and sell it at the same price concurrently, without making losses that can become significant.

Proof of Corollary 1:

If a trader buys and sells the same good, quoted at the same price $B = S$ at the same date, then for all trade from date t_1 onwards one has:

$\pi = N[S(1+R_S)^{-1} - B] = NB[(1+R_S)^{-1} - 1] = -NBR_S(1+R_S)^{-1} < 0$. In a non-negligible period, N can be infinite, given that B is significant and R_S is a positive infinitesimal, this leads to a non-negligible loss in a non-negligible period of continuous trading as $Lim_{N \to \infty} \pi = \text{a negative non-negligible number}$. This reflects the individual's failure to recover the cost of *capital tied-up in transit*, over a non-negligible period.

Thus, she needs a non-negligible "free lunch" to finance this loss in every round of trade. Moreover, these losses can be any significant finite number when N is an infinite natural number. Thus, in the absence of any "free lunch" this loss can become significant, and with unceasing losses, the trader becomes eventually bankrupt. *QED*

Corollary 2:

It is impossible in a risk free "frictionless" market over a significant period of continuous trading to buy a scarce private good at one price and sell it concurrently at a price that exceeds the buying price by a significant amount.

Proof of Corollary 2:

A positive significant number exceeds any infinitesimal. If S exceeds B by a positive significant amount such that $S(1+R_S)^{-1} - B$ is a significant positive number, as N can be infinite in a significant period then $Lim_{N \to \infty} \pi = \text{an infinite positive number}$. That is to say this trade generates an infinite amount of money, hence such a trading opportunity must have an infinite price, and i.e. it is a "money pump", which is impossible. *QED*

Corollary 3:

Under the assumptions of the foregoing theorem, for the same trader the lending rate must exceed the borrowing rate at the same date by an infinitesimal positive number, if she is to engage in both these actions in respect of the same scarce private good concurrently.

Proof of Corollary 3:

From the perspective of the same trader, the selling price of a loan contract as a scarce private good in its own right like any other good must exceed its buying price by a positive infinitesimal at the same date in a risk free "frictionless" market. Hence, the lending rate must exceed the borrowing rate by a *positive infinitesimal* at the same date for the same trader engaged in both these actions in the same instance. *QED*

Lemma 1:

A perfectly matched portfolio of receivable and payable amounts is not a zero-priced hedge, as long as there is a positive instantaneous rate of investment return on any positive amount of money invested.

Proof of Lemma 1:

Let us note that the ownership of a receipt must take at least an instance to become valid, whilst ceasing ownership represented by a payment occurs at a date. Thus, one must receive this amount first before paying it. However, a *perfectly matched* portfolio of receivable and payable amounts represent *concurrent* actions, hence, the payer must find other sources of finance to relieve her obligation timeously, which will not be cost free. The impact of this will be infinitesimal for a finite number of such actions, but it will be non-negligible over a non-negligible period of receiving and paying such amounts continuously, as in continuous-time finance theory. *QED*

Lemma 2:

The profit form the concurrent buying and selling of a scarce private good with volatile prices is risky, when the selling price does not exceed the buying price by 100%, even when there is not a positive rate of investment return on an amount of money invested during such trades.

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Proof of Lemma 2:

Let us note that the price of such a good at each prospective date in an instance can fall, and it is not possible to say that the size of this fall is infinitesimal, unlike the financing cost referred to in lemma 1. On the other hand, for a trader with no prior ownership of that good, it takes at least an instance to acquire it at the price ruling at the beginning of this instance. However, by the end date of that instance, its selling price may fall below its cost of purchase, leading to a loss in present value terms. It thus becomes impossible for this trader to have a risk free profit from concurrent trading in a scarce private good with volatile prices, unless *t*he selling price exceeds the buying price by 100%. *QED*

Lemma 3:

For scarce private goods with volatile prices, there can be a finite number of dates, from the present date onwards, when the selling price exceeds the buying price significantly, but by less than a

100%, leading to the possibility of the realisation of significant economic profits for risk takers.

Proof of Lemma 3:

By assumption, all traders seek to maximise their forecast profits, whilst arbitrageurs seek to minimise the size of this profit. If one assumes the spread between the buying and selling price at each date becomes so significant, such that the profit from this exercise within an instance is significant, then the forecast profit over a significant period from the arbitrage opportunity arising will be infinite as in Corollary 2. However, scarcity of resources implies that this forecast profit will never materialise, hence this spread *cannot* remain significant over a significant period, and arbitrage will reduce it to an infinitesimal minimum within a finite number of instances.

Nonetheless, for scarce private goods with volatile prices, there can still be a *finite* number of dates, including and since the present date, when the selling price exceeds the buying price at the same date significantly, but by less than a 100%. This leads to a finite number of times when the forecast profit is infinite. However, given scarcity of resources and that this forecast profit is not sure, there is no chance of the realisation of a "money pump", but there is the possibility of the realisation of economic profits on each such occasion. A trader will naturally seek to trade on such occasions as far as possible.

It follows that a trader has always the possibility of significant amount of economic profits in a risky market, reflecting the reward for her risk taking. The standard paradigm cannot explain how economic profit arises in a "frictionless" market. QED

4. Explaining the Existence of Corporate Control Premium

The standard paradigm fails to explain the rationale for the existence of corporate control premium. Taking the simple case of an all-equity firm, based on the new foundation in Section 3, the following theorem resolves this problem.

Theorem:

The take over value of an all-equity firm exceeds its flotation value by its capital tied-up in transit.

Definitions, assumptions and their justification:

In exactly the same way that a farmer needs to sow the same kind of seeds that she reaps, a firm needs financial capital i.e. money as input to produce the same as output. However, as this is a "seed" that needs to be *continuously* ploughed into the business, it leads to the requirement for the firm to maintain a certain amount of capital invested *permanently* to continue its operations without any interruption as a solvent concern.

For the all-equity firm studied here, at its inception, this capital represents the initial investment outlay for all non-monetary and monetary inputs in one input-output reproduction cycle. Thereafter, it can be augmented by the retained economic profits undistributed as dividends, and reinvested in the firm to expand the business; hereafter the price of the entire investment for all inputs in one input-output reproduction cycle is called *Locked-in-Equity* (LE). (It is pointed out in Lemma 3 that in *risky markets* significant amounts of economic profits, and hence dividends, can be generated. In such a market, LE can also include a *reserve* i.e. a relatively "safe" investment against uninsurable risk e.g. unforeseeable calamities.)

The concept of LE for a firm is very much akin to the concept of *capital tied-up in transit* for a purchaser of a scarce private good in Section 3. In this case, LE is the financial capital i.e. the money representing the buying price of all the inputs of the firm consumed in one input-output reproduction cycle. For a new firm, it can be seen as the start-up cost for one cycle of reproduction. The standard paradigm overlooks the existence of LE.

Let us now make a distinction between *ownership* and c*ontrol.* Section 3 defined ownership as follows. An individual is said to *own* a positive quantity of an object, if she can have immediate, direct and exclusive *access to it for any purpose offered by the object throughout her ownership duration*. This means that the good is used in that duration precisely for *any* possible purpose that she decides for it. *Control* over a scarce private good can be defined as giving the *controller* the right of immediate, direct and exclusive access to the good *only for the purposes specified by the owner in a contract with the controller.* Obviously, the specific purposes the owner decides for the controllers will be only a small subset of all the purposes available to the owner. *The cost of managing a company is assumed to be the same irrespective of who controls it.*

In standard portfolio theory, diversification reduces investment "risk", in the way the latter concept is understood in the standard paradigm. However, it was noted in Section 3 that "risk" in there is the possibility of a gain *or* loss, moreover, "risk" as such overlooks the impact of control (or lack of it) on the investment. Hence, there is no compelling reason to recommend diversification as the *only* "rational" choice in a risky or risk free market.

Now consider a situation where there are only two mutually exclusive groups of investors in companies: one group of investors called the *generalists* prefer to have a diversified portfolio of investment in companies, and another group called the *specialists* prefer to specialise in the investment of a single company. *Each* specialist investor owns *all* the shares of the investee company, whilst it takes *very many* generalist investors to own a single company, such that no single generalist owns sufficient shares with voting rights to impose her will over the investee company.

The generalists at the inception of the company appoint its *controllers,* whose contracts pass on to the controllers the collective right of the *generalist* shareholders to control the company. Thereafter, the generalists let the company be run by its controllers, as long as they abide by their contracts with the shareholders. The latter is taken for granted here. The controllers also choose their own successors. Thus, the generalists do not co-ordinate and organise the collective control of their group of investors over the company as a single body, by assumption.

However, the generalist shareholders retain their right of buying and selling their shares individually, from and to, any investor. Thus, they "vote" by their individual actions of buying or selling shares. That is done in response to news about the company or changes in investors' preferences, or their liquidity needs and so on. In particular, there is nothing to stop the generalist shareholders to sell their shares to a *specialist* investor, in which case, *the generalist shareholders' contract with the controllers will terminate by assumption.* In contrast to the generalist, the specialist investor retains *full control* of the company for herself, although she may use others to help her with this job.

By definition, whilst individual generalists own *shares* in the company, after its inception they do not own the *company*; as by assumption, they cannot have immediate, direct and exclusive access to the resources of the company for *any* purpose they wish. The *flotation value* of a company can thus be defined as the buying price of its *shares,* as objects in their own right, from the perspective of individual generalist investors in that company. In contrast, a specialist fully owns and controls her company. The specialist has *total* freedom on her investment e.g. she can implement her ideas in the running of the company and if she so chooses, she can cease all its operations and release its LE. Thus, the *take over value* of the company represents the buying price of the *company* in its entirety from the perspective of a specialist investor.

Proof of the Theorem:

The flotation value of a company's shares, from the perspective of generalist investors, *does not* reflect the buying price of control over it by definition. This is because these generalist shareholders have released control of the company to its *controllers*. Hence, they do not have access to the company's internal resources by assumption. In contrast, when a floated company is taken over by a single specialist investor, who ends up with full control over the company, LE will then be accessible to this acquirer. Thus, the value of LE *to the bidder* will be reflected in the take over price of the company, and realised in the form of *corporate control premium*.

If the firm is to generate dividends indefinitely, as a going concern, it should also retain a LE at all times indefinitely to make this possible. However, the generalists never can find access to this LE as long as the controllers abide by their contracts with the shareholders, whilst the specialist owner of *the company, as a going concern, has full access to it at all times by definition.*

Indeed, one might say that generalists are buying a stream of future dividends, whilst the specialists are buying not only those dividends but also the retained financial capital of the company, represented by its prospective LE. Looking at it from another angle, the specialist investor as a new entrant to the industry will require a LE as the start-up cost of running the acquired business, and thus has to pay the buying price of the LE of that company. The error in the standard paradigm is that it is implicitly assumed that all the retained capital of the company, *including its capital tied-up in transit*, is eventually paid out as dividends. However, without the *capital tied-up in transit*, the firm would cease to exist.

This paper points out (under Section 3) that the buying price of any investment is the present value of the investor's prospective *accessible* cash flows from the date this access is obtained. Thus, the dividend valuation models yield the flotation value of a company in the hands of generalist shareholders as individuals, rather than a collective united body, and properly ignore the retention of the firm whilst the shareholders are *not* collectively controlling the company as a single body. On the other hand, by the same present value criterion, the take over value of the firm exceeds its flotation value by the value of the company's prospective *LE* to the bidder, who gains access to it. Thus, corporate control premium reflects the buying price of the prospective *Locked-in-Equity* of a company.

For example, if the business of the trader in the Proof part of Section 3 were listed in a stock market as a non-dividend paying company, its flotation value at t_N where $N > 1$ would be zero. However, its take over value is significant, and it reflects the value of her locked-in equity in this business i.e. the *total capital tied-up in transit*. This is equal to $2B$, in terms of prices at t_1, if she were to start up another business like it or $2S$ if she were to exit it.

Assuming the same business was operating in a risky market such as that described in Lemma 3, and distributed all its economic profits as dividends, its flotation value will then be the present value of those dividends. However, its take over value will reflect the value of the locked-in equity in this business i.e. the total capital tied-up in transit, as a premium over its flotation value. This analysis implies that if a specialist investor wants to sell her company, she must try to sell it to another *specialist* investor, as far as possible, and it is only after exhausting this avenue that she may consider flotation of the company.

Now that the gist of the issue is resolved, it is possible to relax some of the assumptions to make them more realistic, and note that the conclusion remains intact. In particular, regarding market participants' investment strategy, this assumption can be relaxed so as the same participant can be a specialist investor in respect of one company and a generalist in respect of other companies. However, obviously she cannot be a specialist and a generalist investor in the same company concurrently.

Moreover, over a period of time a specialist in one company may become a generalist in it (and vice versa) due to changes in attitude, individual's circumstance or company characteristics. This can reflect the *conditional and changing* nature of an investor's preferences over the ownership of the company relative to other goods throughout her life e.g. youth versus retirement. The latter will impact the investor's *own* valuation of the *company* versus its *shares*. This explains why a specialist owner of a company may gradually float off her stake, if she and her heirs lose interest in controlling it. *QED*

Distinction between debt and equity

The equity holders of a firm, as a collectively united body, have the right of control over the company, due to the voting rights offered to shareholders. In contrast, debt holders do not have such a right. On the other hand, whoever maintains control over the *capital tied-up in transit* of a firm controls the company. Thus to retain the right of control over the company, the shareholders must possess its *capital tied-up in transit. This is the key to understanding corporate capital structure and dividend policy, which is completely overlooked by the standard paradigm.*

Other puzzles in finance e.g. regarding discount to net asset value for investment trusts (closed end investment funds) etc. can be resolved based on the recognition of the need to maintain a Locked-in-Equity for the firm, on which, more in the future.

5. Conclusion

One may naively think that a formal model *easily* reveals any of its internal inconsistency. Alas, contradictions can remain hidden even in a formal model (Dummett, 1987), hence the need for an enquiry into the foundational issues in economics, given so many puzzles. At this deep level, this paper unearths a hidden antinomy in the way time is treated in economic theory. This contradiction takes its most clear form for an individual in the identity of the welfare effect of the loss of an own good to another party, with the owner's consumption of the same good, in the standard utility function.

The paper points out that the standard paradigm confuses the concept of a *date* i.e. a point in time as against an *instance* i.e. a positive infinitesimal quantity of time, it thus inadvertently assumes that even an infinite number of consecutive trades take no time. This is equivalent to presupposing that the rate of interest is zero in all instances of continuous trading, i.e. permanently. However, this is in contradiction to the acknowledgement of the existence of a positive rate of interest over time for loans of scarce private goods e.g. money. This paper proves that in a "frictionless" market with a positive instantaneous rate of interest the selling price of a scarce private good must exceed its buying price by a positive infinitesimal amount at the same date, from the perspective of the same trader who engages in both actions concurrently.

This is necessary to prevent bankruptcy on the one hand and a "money pump" on the other, over a non-negligible period. This paper thus provides *logical* support rather than a mysterious psychological motive for certain findings in experimental economics, and throws light on empirical and theoretical "anomalies". It is proved analytically that free arbitrage theory, existing utility theory and standard equilibrium theories all collapse, given scarce resources. Moreover, it disproves subjective probability theory, capital asset pricing model and the dividend irrelevance theorem.

Debreu (1987) notes that: "The very definition of an economic concept is usually marred by a substantial margin of ambiguity". By translating these concepts into mathematical notations with precise conceptual meanings, these ambiguities are hidden on the surface, giving the impression of rigour to the analysis, but they are not eliminated. The problem becomes acute when two opposing concepts have over-lapping "margins" e.g. a date and an instance. Then mathematics contributes to the collapse of such formalisations, in Isaac Newton's words: "In mathematics the minutest errors are not to be neglected".

This paper serves to point out that, to construct a new paradigm on firm foundations, one must exhaust all effort to eliminate ambiguities from economic concepts, and certainly remove all *overlapping* "margins of ambiguity", *before* translating them into non-paradoxical mathematical forms. This paper has started off this process.

The standard "frictionless" paradigm cannot explain why a firm as the acquirer and provider of financial capital exits. Nonetheless, it seeks to explain how the firm reaches equilibrium, manages its finances and how its stocks are evaluated, etc. but it fails. The findings of this paper help begin to revise thoroughly economic and financial theory to explain its age-old puzzles. A new paradigm is introduced which, inter alia, identifies the capital tied-up in transit of a trader. The latter is hidden from the eye of a believer in the standard paradigm, whilst this helps the new paradigm to resolve such deeply elusive puzzles as the reason for the existence of corporate control premium.

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