

Where is the efficient frontier

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October 1999

We thank Lillian Ng, Changyun Wang and Lixin Wu for helpful comments.

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Abstract

Observing that the volatility of short term fixed income securities forms a continuous spectrum, we prove that the efficient frontier of risky assets converges naturally to the capital market line. This shows that the attainable combinations of risky assets contain the whole region under the capital market line and the inclusion of riskless asset merely adds to the border. This unifying approach relieves the need to construct efficient frontiers in many asset allocation problems. More important, by removing the restriction posed by the efficient frontiers, it allows for much better asset allocation decisions than the traditional methods.

Where is the efficient frontier

Constructing efficient frontiers is at the heart of many asset allocation decisions. In the capital asset pricing model, risky assets and riskless asset are two distinct classes. The efficient frontier constructed from the risky assets is always away from the capital market line, which emanates from the riskless point and is tangent to the efficient frontier.

However, there is no chasm between risky assets and riskless asset. Suppose our investment horizon is one year. Then a zero coupon, default free bond that matures in one year is a riskless asset. Another zero coupon, default free bond that matures in one year and one day is a risky asset, for its price one year from now is determined by the overnight rate at that time. The price uncertainty of the bond a year from now will be very small for the short maturity, i.e., one day. Because of the abundance of short term high grade fixed income securities, we can assume the volatility of risky assets forms a continuous spectrum that converges to zero, the volatility of riskless asset. With this observation in mind, we proceed to prove that the efficient frontier of the risky assets converges to the capital market line. That is, for most practical purposes, the capital market line is the efficient frontier for all the risky assets.

In Section I, we give a brief review of the historical works and present our main results. We then use an example to illustrate how our results can significantly improve the asset allocation decisions. Section II concludes.

1. Main results

Markowitz (1952) first described the efficient frontier of the portfolios of risky assets. Sharpe(1964) showed that under equilibrium, the optimal asset allocation is combination of risky assets plus the risk free asset. In these works, the region of attainable combinations of risky assets is distinctly away from the riskless asset. The dichotomy of risky assets and riskless asset becomes the standard in subsequent works. However, this strict dichotomy has not taken into account the properties of the short term fixed income markets.

The short term fixed income markets are extremely liquid. There are money market, exchange traded interest rate futures market, high grade corporate debt market and government debt market. If we set the investment horizon to be T , which is short term, say, less than one year, then the default free, zero coupon fixed income security that matures at time T is a riskless investment. Given the abundance and liquidity of the

fixed income securities that mature near time T , we can make the following assumption.

Assumption 1: The volatility of risky assets forms a continuous spectrum that converges to zero.

The following is another common assumption.

Assumption 2: The expected return of any risky asset is higher than that of the riskless asset.¹

First, we construct a mean-variance efficient locus from two assets, one the market portfolio, M , and another asset with very low volatility, A .

Suppose the covariance matrix of the two assets is

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

where σ_{11} and σ_{22} are the variances of asset A and market portfolio M respectively. The expected return vector of A and M is

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Let σ and μ be the standard deviations and means of the weighted portfolios of A and M . From Ingersoll(1987), the mean-variance efficient locus of A and M is

$$s^2 = \frac{am^2 - 2bm + c}{\Delta} \quad (1)$$

Where

$$\begin{aligned} a &\equiv 1'\Sigma^{-1}1 > 0, \quad b \equiv 1'\Sigma^{-1}Z, \\ c &\equiv Z'\Sigma^{-1}Z, \quad \Delta \equiv ac - b^2 > 0 \end{aligned}$$

Equation (1) is a parabola connecting M and A in the mean-variance space. Now we have the following:

Proposition 1: Suppose $\sigma_{11} < \varepsilon^2$, where σ_{11} is the variance of asset A . In the mean-standard deviation space, the biggest distance from the mean-variance efficient locus of asset A and market portfolio M to the capital market line is less than ε .

Proof: See Figure 1. RM is the capital market line. In the mean-standard deviation space, AB , the distance from A to the mean axis, is less than ε . From Assumption 2, AC , the distance from A to the capital market line, is

less than ε as well. Since RM is tangent to the mean-variance efficient locus at M and the efficient locus is a concave function, the distance from the efficient locus to the capital market line is a decreasing function from A to M .

The following is the main result of the paper.

Proposition 2: The global efficient frontier of all risky assets is the capital market line.

Proof: The global efficient frontier of all risky assets must be at the left side of the mean-variance efficient locus of two risky assets. Apply Proposition 1 and let ε approach zero.

The above result shows that the attainable combinations of risky assets form an open set bordered by the capital market line. The inclusion of riskless asset merely adds the border to the original set. This result is actually very natural after we present it. It means that the riskless asset is the boundary of risky assets.

Proposition 2 shows that for most practical purposes, we can use the capital market line as the global efficient frontier. This has significant

implications in practice. Take an example from Page 105 in Elton and Gruber(1995). The input data for asset allocations are expected returns, standard deviations and correlation coefficients of S&P, bonds, international stocks of Canada, Japan, emerging market, Pacific, Europe and small stocks. Computation gives the following weighting for the global minimum variance portfolio

S&P	Bond	Canadian	Japan	Emerging market	Pacific	Europe	Small stock
-30.19	101.00	7.09	-1.62	-12.96	2.92	15.84	17.93

Notice the weight of -30% in S&P 500 and other negative weights in Japan stocks and emerging market stocks in constructing the global minimum variance portfolio. It is difficult to imagine any one will construct a low variance portfolio with so many short positions on stocks with such high weights. Where does the problem come from?

Proposition 1 and 2 illustrate the importance of short term fixed income securities in constructing an efficient frontier. However, in the above example, all the fixed income securities are grouped under one category, bonds. The underestimation of the importance of the fixed income securities in asset allocation seriously distorts the shape of the efficient frontier, which restricts the selection of optimal portfolios. Figure 2

shows the position of the global minimum variance portfolio, P , when all the fixed income securities are grouped under one category. However, any point on PD , which has the same volatility and higher expected return, can be constructed from the available risky assets. When a portfolio is close enough to D , all the assets have positive weights. This example shows that a sophisticated yet not well specified model often provides a poorer solution than simple intuition.

2. Conclusion

Traditionally, risky assets and riskless asset are treated as two distinct classes. Observing the numerous fixed income securities with very close maturity dates, we view riskless asset as the natural limit of risky assets. This unifying viewpoint is not only theoretically appealing, but also practically important. This work relieves the burden of constructing efficient frontiers in many asset allocation problems. More important, by removing the restriction posed by the efficient frontiers, it allows for much better asset allocation decisions than the traditional methods.

Footnotes:

1. This assumption is only for technical convenience. As long as the return of a risky asset converges to the return of the riskless asset when its volatility converges to zero, all the results will hold.

References

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Figure legends:

Figure 1: The distance from the efficient locus to the capital market line.

Figure 2: Points on DP are attainable combinations of risky assets which have the same risk, but higher expected return than P, the global minimum variance point with respect to a certain efficient frontier.

Figure 1

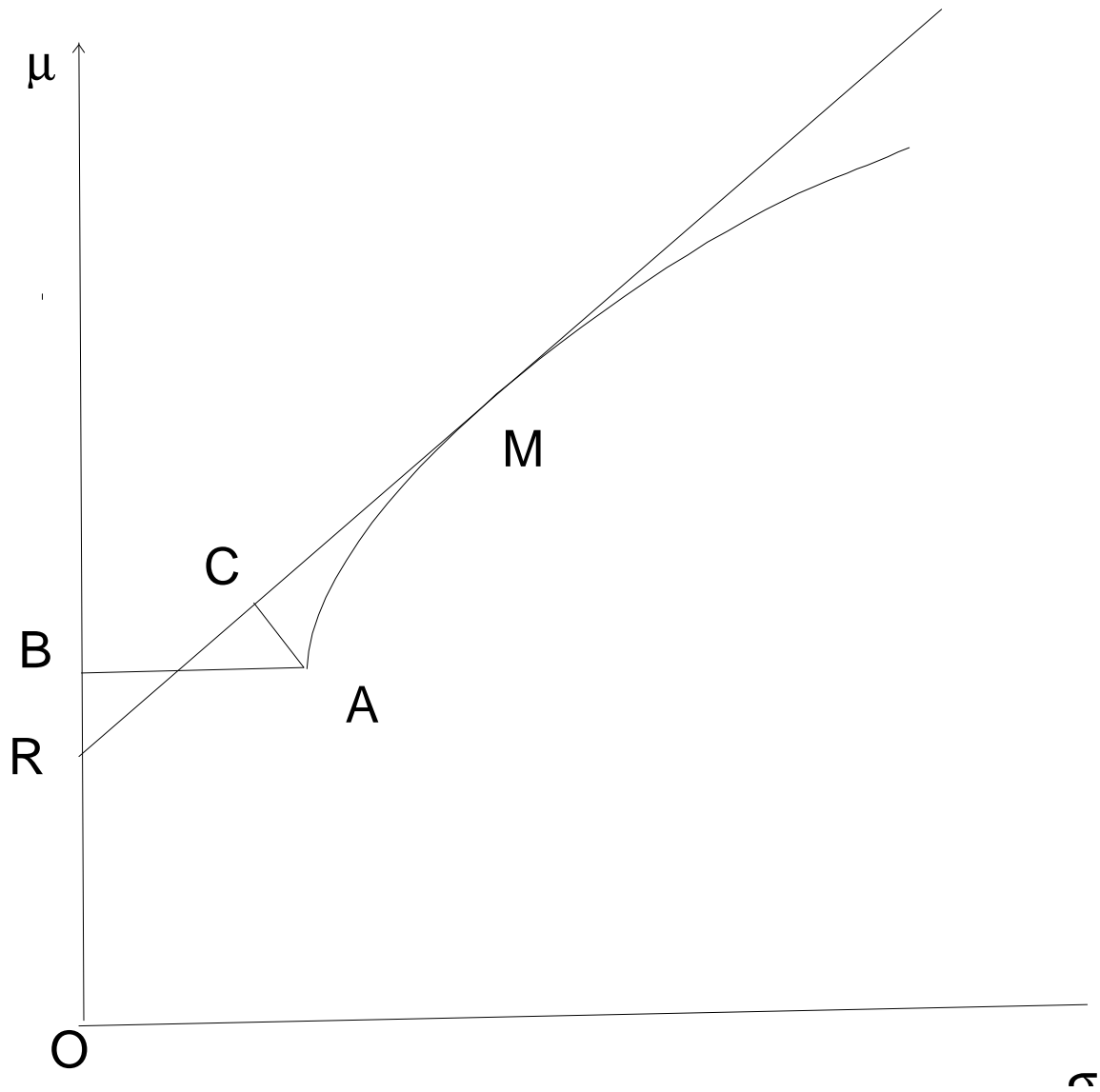


Figure 2

